ISRAEL JOURNAL OF MATHEMATICS 82 (1993), 87–131

AUTOMORPHIC FORMS FOR SUBGROUPS OF THE MODULAR GROUP*

BY

HERSHEL M. FARKAS

Institute of Mathematics
The Hebrew University of Jerusalem, Jerusalem 91904, Israel

AND

IRWIN KRA

Department of Mathematics
State University of New York at Stony Brook, Stony Brook, NY 11794, USA

This paper is dedicated to Prof. J. Thompson
in recognition of his many contributions to mathematics
and of his being the recipient of the Wolf Prize in Mathematics

ABSTRACT

The theory of theta constants with rational characteristics is developed from the point of view of automorphic functions for the principal congruence subgroups of the modular group PSL(2, $\mathbb{Z}$). New identities are derived and particular emphasis is given to the level 3 case where a striking generalization of the classical λ-function is obtained.

* Research by HMF supported in part by the Paul and Gabriella Rosenbaum Foundation and partially sponsored by the Edmund Landau Center for Research in Mathematical Analysis, supported by the Minerva Foundation (Germany). Research by IK supported in part by NSF Grants DMS 9003361 and 9204092.
Received August 30, 1992 and in revised form December 10, 1992

1. Introduction

It is well known that $\Gamma(k)$, the principal congruence subgroup of prime level k is torsion free and has finite index in the modular group Γ. The group $\Gamma \cong \mathrm{PSL}(2,\mathbb{Z})$ is generated by the motions* A: $z \mapsto -1/z$ and B: $z \mapsto z+1$ and acts as a group of complex analytic isomorphisms (Möbius transformations) on the upper half plane $\mathbb{H}^2$. The quotient group $\Gamma/\Gamma(k)$ is the full automorphism group of the surface of finite analytic type $\mathbb{H}^2/\Gamma(k)$. Let $p = p(k)$ be the genus of this surface and $n = n(k)$ the number of punctures on it. It is well known (see for example [2], Chapter I) that

$$p(k) = 0 \text{ for } k = 2$$

and

$$p(k) = 1 + \frac{(k^2-1)(k-6)}{24} \text{ for } k > 2,$$

while

$$n(k) = 3 \text{ for } k = 2$$

and

$$n(k) = \frac{k^2-1}{2} \text{ for } k > 2.$$

The automorphism group must permute the $n(k)$ punctures and this suggests that one should view $\Gamma/\Gamma(k)$ as a subgroup of (represented on) $\mathcal{S}_{n(k)}$, the permutation group on n elements. One of our aims is to study this representation, and its relation to function theory (involving mostly theta constants) for the Riemann surfaces defined by the principal congruence subgroups.

In this introduction, we review the mostly known theory for the case $k = 2$ in order to motivate our subsequent presentation. It is interesting that even in this classical case we seem to obtain some new results. The basic ideas are not new. We define an important equivalence relation on $\mathbb{R}^2$. For reasons that will become clear shortly we consider vectors in $\mathbb{R}^2$ to be **theta characteristics.** Two characteristics

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$$

---

* When we need to lift these elements of PSL(2, **Z**) to SL(2, **Z**), we will identify B with the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and A with $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

are called **equivalent** provided their sum or difference belongs to $2\mathbb{Z}^2$. We denote this equivalence by the symbol $\equiv$.* The quotient of $\mathbb{R}^2$ by this equivalence $\equiv$ (the space of **characteristic classes**) can also be viewed as the orbit space of $\mathbb{R}^2$ by the group of motions generated by the three transformations

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -x \\ -y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+2 \\ y \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y+2 \end{bmatrix}.$$

Using the usual identification of $\mathbb{R}^2$ with $\mathbb{C}$, we see that the space of characteristic classes can be identified with the Riemann surface (orbifold) of signature $(0,4;2,2,2,2)$ corresponding to the elementary group generated by $z \mapsto -z$, $z \mapsto z+2$ and $z \mapsto z + 2\imath$. See [1]; p. 227. We define a right action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$ (viewed as theta characteristics). For the unimodular matrix

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

and the characteristic

$$\chi = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2,$$

we define the characteristic $\chi\gamma$ by the formula

$$\begin{bmatrix} a\epsilon + c\epsilon' - ac \\ b\epsilon + d\epsilon' + bd \end{bmatrix}.$$

This does *not* define a group action of $\mathrm{SL}(2,\mathbb{Z})$ on characteristics. It does however define a (right) group action of $\mathrm{PSL}(2,\mathbb{Z})$ on characteristic classes. We will also need to select finite sets of characteristic classes that are fixed pointwise and/or permuted by the modular group and its principal congruence subgroups. See §3 and §9.

Consider now the set of characteristic classes represented by vectors $\begin{bmatrix} m/2 \\ m'/2 \end{bmatrix}$ with m and m' integers. It is easy (consult §3, where we consider a similar situation: prime $k > 2$) to see that there are precisely ten such points, four of which are points with integer coordinates:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

* We will also use this symbol for the usual congruences involving integers (modulo a prime). This abuse of symbols should not cause any confusion.

There are six additional points in this set:

$$\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

The group $\Gamma(2)$ is generated by the motions

$$B^2\colon z \mapsto z+2, \qquad A \circ B^{-2} \circ A\colon z \mapsto \frac{z}{2z+1}$$

which we denote by S and W, respectively. Each of these (and hence the group $\Gamma(2)$) fixes the four characteristic classes with integer entries and permutes the other six. Each element of Γ fixes $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and permutes the other three integral characteristic classes. It therefore follows that we have a homomorphism η of Γ into the permutation group on three elements. In order to use the usual cycle notation for permutations, we identify characteristic classes with integers as follows:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \sim 1, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sim 2, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sim 3.$$

Using the generators of Γ introduced above, we find that B induces the permutation, written in cycle notation, $(1\ 2)$ and A induces the permutation $(2\ 3)$. These two permutations generate the full permutation group $\mathcal{S}_3$. Since, as we have already mentioned, the subgroup $\Gamma(2)$ is the kernel of the homomorphism η, we get the well known result that $\Gamma/\Gamma(2)$ is isomorphic to $\mathcal{S}_3$.

Let us consider next the action of $\Gamma(2)$ on the set Y consisting of the six additional characteristic classes we have described above. Let us denote them (in analogy to what we did to integral classes) by:

$$\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \sim 1, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \sim 2, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \sim 3, \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \sim 4, \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \sim 5, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \sim 6.$$

We find that S induces the permutation $(2\ 6)(3\ 4)$ and that W induces the permutation $(1\ 5)(2\ 6)$. These generate the Klein 4-group, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The quotient Y modulo the Klein 4-group consists of the three orbits

$$[1,1,5,5], \quad [2,2,6,6], \quad \text{and} \quad [3,3,4,4]$$

(we have written each element according to its multiplicity). The group Γ acts on these orbits: B fixes the first orbit and permutes the second and third, while A fixes the third and permutes the first and second. If we now compare the actions of Γ on these orbits and the action of Γ on the integral characteristic classes, it is clear that we should associate

$$\text{the characteristic class of } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ with the orbit } [3,3,4,4],$$

$$\text{the characteristic class of } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ with the orbit } [2,2,6,6],$$

and

$$\text{the characteristic class of } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ with the orbit } [1,1,5,5].$$

The above is a special case of a phenomenon which we shall investigate also for the case of $k = 3$ (see §10). The study of the general case ($k > 3$) is deferred to a subsequent paper.

With the above association, the action of Γ on the orbits is the same as the action of Γ on the three even integer characteristics. The orbits give a pairing of the six theta characteristics and ultimately yield the equations (see §9)

$$\frac{\theta^2\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}{\theta^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \frac{1}{2}, \qquad \frac{\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\theta^2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta^2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \frac{\imath}{2},$$

$$\text{and} \qquad \frac{\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}\theta^2\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = -\frac{1}{2}.$$

(The fourth power of the above ratios is constant, 1/16.)

It is known that $\mathbb{H}^2/\Gamma(2)$ is a three times punctured sphere and that the function

$$f\colon \tau \mapsto \frac{\theta^4\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)}{\theta^4\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau)}$$

is holomorphic on $\mathbb{H}^2$, automorphic with respect to $\Gamma(2)$ and maps $\mathbb{H}^2$ onto the Riemann sphere less the three points 0, 1, ∞. In fact, f is a holomorphic universal cover of $\mathbb{C} - \{0,\ 1\}$. There are many ways to obtain this result. The one relevant to us and the one we shall generalize (to some extent in this paper and more in subsequent work) is the following. It can be shown that for $\gamma \in \Gamma(2)$, each of the three even integral theta constants (that is, ϵ and ϵ' are each zero or one and $\epsilon\epsilon' = 0$) satisfy the relation

$$\theta^4 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \gamma(\tau))\gamma'(\tau) = \pm\theta^4 \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \tau)$$

(for all $\tau \in \mathbb{H}^2$; where the choice of sign depends only on the motion γ) so that f projects to a meromorphic (perhaps multivalued) function F on the Riemann surface $\mathbb{H}^2/\Gamma(2)$, a thrice punctured sphere. The function f is holomorphic and non-zero on $\mathbb{H}^2$ and has a well defined limit as $\tau \in \mathbb{H}^2$ tends to any $r \in \mathbb{Q} \cup \{\infty\}$ through a cusped region belonging to r. Easy calculations (see §8) show that

$$f(\infty) = 0, \qquad f(0) = 1, \qquad f(1) = \infty.$$

It is a simple exercise to show that the projection of each of the fourth powers of the three classical theta constants to $\mathbb{H}^2/\Gamma(2)$ (a Prym differential on the thrice punctured sphere) has simple poles at two of the punctures and is bounded at the third. Further, distinct theta characteristics lead to differentials with distinct sets of poles. The quotient of two of these therefore gives rise to an analytic homeomorphism of $\mathbb{H}^2/\Gamma(2)$ onto the sphere punctured at the points 0, ∞ and a third point. We already know that this third point must be 1. (We conclude that F is single valued because a multivalued function must have more complicated singularities than those possessed by F.) It follows that

$$\theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \neq \theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \quad \text{for all } \tau \in \mathbb{H}^2,$$

a fact which is well known but not obvious.

In this paper we shall show how to extend many of the results discussed above to the other principal subgroups of the modular group with particular attention to the cases $k = 3$ and 5.

This paper lays the foundation for continuing work. Much of the material on the combinatorics of punctures and characteristic classes (most of the material of sections 2 through 6) is not needed for the theory in the rest of this paper. It will be used in subsequent papers. We thank the referee for pointing out some applications of our work on cusp forms (in particular, of Theorem 3) to number theory (the growth as $n \to \infty$ of the number of solutions to $Q(x_1, \ldots, x_{8k}) = n$, where Q is a quadratic form in $8k$ variables that depends on the characteristic $\left[\begin{smallmatrix} m/k \\ m'/k \end{smallmatrix}\right]$) which we will present in a short independent note.

2. The quotient surface $\mathbb{H}^2/\Gamma(k)$

As remarked in the introduction the quotient $\mathbb{H}^2/\Gamma(2)$ is a thrice punctured sphere while for a prime $k > 2$, $\mathbb{H}^2/\Gamma(k)$ is of type $(1 + \frac{(k^2-1)(k-6)}{24}, \frac{(k^2-1)}{2})$. It is easily seen that ∞, 0, and 1 are a (maximal) set of inequivalent (parabolic) fixed points for $\Gamma(2)$. According to Chapter I of [5], for the odd prime k, one obtains a maximal set $\mathcal{Q}$ of inequivalent parabolic fixed points by following the recipe given below. Identify $a/b \in \mathbb{Q} \cup \{\infty\}$ with the vector

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$$

(as usual identify $a/0$, $a \neq 0$, with ∞). Consider only those pairs a and b of relatively prime integers. Infinity is a parabolic fixed point of the group $\Gamma(k)$. We use it as one of our parabolic fixed points. A cusp $x \in \mathbb{R}$ is $\Gamma(k)$-equivalent to ∞ if and only if $x = a/b \in \mathbb{Q}$ with $a \equiv \pm 1 \bmod k$ and $b \neq 0$, $b \equiv 0 \bmod k$. Let X be the subset of $\mathbb{R}^2$ consisting of vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with x and $y \in \mathbb{Z}$ relatively prime. The points of X determine all the cusps for the group $\Gamma(k)$. The vectors

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad z' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

in X correspond to the same puncture (are $\Gamma(k)$-equivalent cusps) if and only if $z' \equiv \pm z \bmod k$. A set of representatives for this equivalence relation on X produces $\mathcal{Q}$. A convenient list for the points is provided by the following table consisting of $(k+3)/2$ rows. The first row consists of the vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. The second row consists of the vectors $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$, $i = 0, 1, \ldots, k-1$. For $j = 3, \ldots, (k+1)/2$, the j-th row consists of k vectors of the form $\left[\begin{smallmatrix} m_i \\ j-1 \end{smallmatrix}\right]$ where for $i = 1, \ldots, k$, m_i an integer relatively prime to $j-1$ and $\equiv i \bmod k$. (For example, the m_i in the third row, assuming $k > 3$, are

$$1,\ k+2,\ 3,\ k+4,\ 5,\ k+6,\ \ldots,\ k-2,\ 2k-1,\ k.)$$

The last row consists of the $(k-3)/2$ vectors $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]$ with $m \in \mathbb{Z}$, $2 \leq m \leq (k-1)/2$. It is clear the above list produces $1 + \frac{k-1}{2}k + \frac{k-3}{2} = \frac{k^2-1}{2}$ inequivalent parabolic fixed points. We proceed to another list of such points that will be useful later.

This new list (to be called the **list of punctures**) will consist of $k+1$ rows; each row will contain $(k-1)/2$ entries. The first row of this new list is

$$\begin{bmatrix} 1 \\ k \end{bmatrix}, \begin{bmatrix} 2 \\ k \end{bmatrix}, \begin{bmatrix} 3 \\ k \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-1}{2} \\ k \end{bmatrix}.$$

The above row is the complete set of representatives of the punctures on $\mathbb{H}^2/\Gamma(k)$ that are fixed by the automorphism, $\tilde{B}$, of this surface induced by the Möbius transformation B: $z \mapsto z+1$. First, it is clear that each of these punctures is fixed by $\tilde{B}$. Conversely, if the puncture corresponding to the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $y \neq 0$ is fixed by $\tilde{B}$, then

$$\begin{bmatrix} x+y \\ y \end{bmatrix} \equiv \pm \begin{bmatrix} x \\ y \end{bmatrix} \bmod k.$$

It follows that $y = km$ for some nonzero $m \in \mathbb{Z}$. Without loss of generality $m = 1$. It follows that we may also assume that $1 \leq x \leq (k-1)/2$. We have shown that $\tilde{B}$ fixes precisely $(k-1)/2$ of the $n(k)$ punctures on $\mathbb{H}^2/\Gamma(k)$.

The second row of our new list is the image of the first row under the motion A: $z \mapsto -1/z$. For $j = 3, \ldots, k+1$, the j-th row is the image of the $(j-1)$-st row under the map B. We see that the elements of the j-th row ($j = 2, 3, \ldots, k+1$) are the cusps

$$\begin{bmatrix} k+(j-2) \\ 1 \end{bmatrix}, \begin{bmatrix} k+2(j-2) \\ 2 \end{bmatrix}, \begin{bmatrix} k+3(j-2) \\ 3 \end{bmatrix}, \ldots, \begin{bmatrix} k+\frac{k-1}{2}(j-2) \\ \frac{k-1}{2} \end{bmatrix}.$$

It is routine (but a bit tedious) to establish one-to-one correspondences between the punctures represented in the two lists. We should remark that by what we already established, it is obvious that the punctures represented by the cusps in the j-th row ($j = 2, \ldots, k+1$) of our second list are precisely the punctures on $\mathbb{H}^2/\Gamma(k)$ fixed by the automorphism of this surface induced by the Möbius transformation $B^{j-2} \circ A \circ B \circ A^{-1} \circ B^{2-j}$ (a parabolic motion with fixed point at $j-2$).

Since $\tilde{B}$ is of prime order k, the branch number at $x \in S = \mathbb{H}^2/\Gamma(k)$ of the natural projection $S \to S/<\tilde{B}>$ is $k-1$ at each fixed point of $\tilde{B}$. If such a fixed point existed we would be able to find a lift C of $\tilde{B}$ to $\mathbb{H}^2$ which would be elliptic of order k. This motion C would be an element of Γ; which would imply that $k = 2$ or 3. When $k = 2$ or 3, the surface $\mathbb{H}^2/\Gamma(k)$ is a sphere; which implies that $\tilde{B}$ has precisely two fixed points. Exactly one of the punctures is fixed. Hence one interior point must also be fixed. We thus conclude by Riemann–Hurwitz that for prime $k > 3$, the genus of $S/ < \tilde{B} >$ is

$$1 + \frac{(k-1)(k-11)}{24}.$$

In particular, for $k = 5$ and 7 the quotient is a sphere (also, of course, for $k = 2$ and 3).

It is a consequence of the above that the Riemann surface S factored by the group of automorphisms generated by $\tilde{B}$ has precisely $k-1$ punctures. It is clearly the Riemann surface obtained by factoring the upper half plane by the group $\Gamma(k)$ with the motion B adjoined.

For example, if $k = 5$, the surface S is a 12 times punctured sphere and the quotient is a 4 times punctured sphere. We will show in the sequel that an explicit map of the one to the other is given by the fifth powers of a quotient of theta constants. Now any such pair which gives a function of order five will do but there is a semi-canonical choice, the quotient

$$f = \frac{\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}},$$

which has the advantage of being invariant under $\tau \mapsto \tau + 1$. So, for example, f is a function on the quotient surface. The cases $k = 2$, 3 can also be handled in this way. In fact, in the case $k = 2$ the map is given by $\lambda^2/(\lambda - 1)$, where λ is the quotient of fourth powers discussed in the introduction. The case $k = 3$ is similar and will be discussed in §10.

3. Characteristic classes

We have defined in the introduction the space of characteristic classes as the quotient of $\mathbb{R}^2$ by a group of rigid motions. A convenient fundamental domain for this group action is

$$\{(x,y) \in \mathbb{R}^2;\ x > 0,\ y > 0 \text{ and } x + y < 2\} \cup \{(0,y) \in \mathbb{R}^2;\ 0 \le y \le 1\}$$
$$\cup \{(x,0) \in \mathbb{R}^2;\ 0 < x \le 1\} \cup \{(x, 2-x) \in \mathbb{R}^2;\ 0 < x \le 1\}.$$

Another convenient fundamental domain (which will hereafter be denoted by $\mathcal{P}$) for this group action is

$$\{(x,y) \in \mathbb{R}^2;\ 0 < x < 1,\ 0 < y < 2\} \cup \{(0,y) \in \mathbb{R}^2; 0 \le y \le 1\}$$

$$\cup\{(x,0)\in\mathbb{R}^2;\ 0<x\ \le 1\}\cup\{(1,y)\in\mathbb{R}^2;\ 0<y\ \le 1\}.$$

Fix an odd prime k. We are interested in characteristic classes represented by vectors of the form $\begin{bmatrix} m/k \\ m'/k \end{bmatrix}$ with m and m' odd integers. We eliminate from our list the (equivalence class of the) characteristic $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (since the corresponding theta constant vanishes identically and this characteristic is invariant under the entire modular group). Up to equivalence there are $(k^2-1)/2$ such characteristics. We may choose as representatives for these classes, the characteristics

$$\begin{bmatrix} 1 \\ \frac{1}{k} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{3}{k} \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ \frac{k-2}{k} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{k} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{k} \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-2}{k} \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \end{bmatrix}, \begin{bmatrix} \frac{3}{k} \\ \frac{1}{k} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-2}{k} \\ \frac{1}{k} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{k} \\ \frac{3}{k} \end{bmatrix}, \begin{bmatrix} \frac{3}{k} \\ \frac{3}{k} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-2}{k} \\ \frac{3}{k} \end{bmatrix},$$

$$\cdots\cdots\cdots\cdot$$

$$\begin{bmatrix} \frac{1}{k} \\ \frac{k-2}{k} \end{bmatrix}, \begin{bmatrix} \frac{3}{k} \\ \frac{k-2}{k} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-2}{k} \\ \frac{k-2}{k} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{k} \\ \frac{k+2}{k} \end{bmatrix}, \begin{bmatrix} \frac{3}{k} \\ \frac{k+2}{k} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-2}{k} \\ \frac{k+2}{k} \end{bmatrix},$$

$$\cdots\cdots\cdots\cdot$$

$$\begin{bmatrix} \frac{1}{k} \\ \frac{2k-1}{k} \end{bmatrix}, \begin{bmatrix} \frac{3}{k} \\ \frac{2k-1}{k} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{k-2}{k} \\ \frac{2k-1}{k} \end{bmatrix}.$$

In terms of the second fundamental region $\mathcal{P}$ described above the first row of the above list are characteristics on the vertical line $x = 1$ and the remaining rows are on the horizontal lines $y = 1$, $y = 1/k$, $y = 3/k$, $\ldots$, $y = (k-2)/k$, $y = (k+2)/k$, $\ldots$, $y = (2k-1)/k$. They also have the following interesting and important algebraic interpretation: The first row is the set of representatives of those characteristic classes which are invariant (kept fixed) under the element B of Γ (see §5). The second row is the image of the first row under the map A and the collection of elements in the subsequent rows are the images of the second row under the group generated by the motion B. As a matter of fact, this group permutes the columns of the array obtained by excluding the first row of the above table.

It is easy to obtain another useful array by rearranging the one described above. As usual for $x \in \mathbb{R} \cup \{\infty\}$, let Γ_x denote the stabilizer of x in Γ. (This group is infinite cyclic for rational x and trivial otherwise.) Then our new array consists, as above, of $k+1$ rows (each with $(k-1)/2$ elements). The first row consists of those characteristic classes that are fixed pointwise by Γ_∞; while the elements in the j-th row are fixed by Γ_{j-2} for $j = 2, \ldots, k+1$. We call this array the **list of characteristic classes.**

PROPOSITION 1: *The action of the unimodular matrices on characteristics defines a group action of* Γ *on characteristic classes.*

Proof: The result is known (see, for example, Chapter III of [4]). A proof is included for the convenience of the reader. We write the (affine) action of $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ on the characteristic $v \in \mathbb{R}^2$ as a linear transformation followed by a translation

$$v\gamma = \gamma^t v + T_\gamma.$$

(As usual γ^t is the transpose of the matrix γ and for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $ad - bc = 1$, $T_\gamma = \left[\begin{smallmatrix} -ac \\ bd \end{smallmatrix}\right]$.) We first show that the class of $v\gamma$ depends only on the class of the characteristic v and the Möbius transformation determined by the matrix γ. We note that for all real ϵ and ϵ', we have

$$\begin{bmatrix} \epsilon + 2 \\ \epsilon' \end{bmatrix} \gamma = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \gamma + 2 \begin{bmatrix} a \\ b \end{bmatrix},$$

$$\begin{bmatrix} \epsilon \\ \epsilon' + 2 \end{bmatrix} \gamma = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \gamma + 2 \begin{bmatrix} c \\ d \end{bmatrix},$$

$$(-v)\gamma = -v\gamma + 2 \begin{bmatrix} -ac \\ bd \end{bmatrix},$$

and

$$v(-\gamma) = -v\gamma + 2 \begin{bmatrix} -ac \\ bd \end{bmatrix}.$$

We show next that we have a group action of $\mathrm{PSL}(2,\mathbb{Z})$ on characteristic classes. For γ_1 and γ_2 in $\mathrm{SL}(2,\mathbb{Z})$, we have

$$v(\gamma_1\gamma_2) = (\gamma_1\gamma_2)^t v + T_{\gamma_1\gamma_2}$$

and

$$(v\gamma_1)\gamma_2 = \gamma_2^t(\gamma_1^t v + T_{\gamma_1}) + T_{\gamma_2} = \gamma_2^t\gamma_1^t v + \gamma_2^t T_{\gamma_1} + T_{\gamma_2}.$$

Thus to have a (right) group action it suffices to show that

$$T_{\gamma_1\gamma_2} - (\gamma_2^t T_{\gamma_1} + T_{\gamma_2}) \in 2\mathbb{Z}^2.$$

Writing

$$\gamma_1 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad \gamma_2 = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

we need to study the difference between

$$\begin{bmatrix} -(a_1b_1 + a_2b_3)(a_3b_1 + a_4b_3) \\ (a_1b_2 + a_2b_4)(a_3b_2 + a_4b_4) \end{bmatrix}$$
$$= \begin{bmatrix} -(a_1a_3b_1^2 + a_1a_4b_1b_3 + a_2a_3b_1b_3 + a_2a_4b_3^2) \\ a_1a_3b_2^2 + a_1a_4b_2b_4 + a_2a_3b_2b_4 + a_2a_4b_4^2 \end{bmatrix}$$

and

$$\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\begin{bmatrix} -a_1a_3 \\ a_2a_4 \end{bmatrix} + \begin{bmatrix} -b_1b_3 \\ b_2b_4 \end{bmatrix} = \begin{bmatrix} -a_1a_3b_1 + a_2a_4b_3 - b_1b_3 \\ -a_1a_3b_2 + a_2a_4b_4 + b_2b_4 \end{bmatrix}.$$

We will show that the last two characteristics differ by twice an integral characteristic. Since signs are irrelevant mod 2, it suffices to show that

$$a_1a_3b_1 + a_2a_4b_3 + b_1b_3 \equiv a_1a_3b_1^2 + a_1a_4b_1b_3 + a_2a_3b_1b_3 + a_2a_4b_3^2 \pmod 2$$

and

$$a_1a_3b_2 + a_2a_4b_4 + b_2b_4 \equiv a_1a_3b_2^2 + a_1a_4b_2b_4 + a_2a_3b_2b_4 + a_2a_4b_4^2 \pmod 2.$$

But

$$a_1a_3b_1^2 + a_1a_4b_1b_3 + a_2a_3b_1b_3 + a_2a_4b_3^2 = a_1a_3b_1^2 + b_1b_3(a_1a_4 + a_2a_3) + a_2a_4b_3^2$$
$$\equiv a_1a_3b_1^2 + b_1b_3(a_1a_4 - a_2a_3) + a_2a_4b_3^2 \pmod 2 = a_1a_3b_1^2 + b_1b_3 + a_2a_4b_3^2.$$

We are hence reduced to showing that

$$a_1a_3b_1 + a_2a_4b_3 \equiv a_1a_3b_1^2 + a_2a_4b_3^2 \pmod 2$$

This last equation is obvious since

$$a_1a_3b_1(1 - b_1) + a_2a_4b_3(1 - b_3)$$

is the sum of two even integers. Similarly, for the second entries of our theta characteristics. ∎

Remark 1: The above proof relies extensively on the fact that we are dealing with matrices in SL$(2,\mathbb{Z})$ and does not work for matrices in SL$(2,\mathbb{R})$. ∎

4. Characteristics and punctures

In this section we give a procedure for identifying certain sets of characteristic classes with the punctures on $\mathbb{H}^2/\Gamma(k)$.

Definition 1: For the (positive) odd prime k we let

$$A_k = \begin{bmatrix} -2 & -1 \\ k & (k-1)/2 \end{bmatrix}. \quad \blacksquare$$

The matrix $A_k \in \mathrm{SL}(2,\mathbb{Z})$ has the following property (equalities are to be interpreted for characteristic classes): Let $n \in \mathbb{Z}$. For $k = 4l+1$,

$$\begin{bmatrix} 1 \\ n/k \end{bmatrix} A_k = \begin{bmatrix} 1 \\ \frac{\frac{1}{2}(k+1)}{k} n \end{bmatrix}$$

and for $k = 4l+3$,

$$\begin{bmatrix} 1 \\ n/k \end{bmatrix} A_k = \begin{bmatrix} 1 \\ \frac{\frac{1}{2}(k-1)}{k} n \end{bmatrix}.$$

In particular, it follows that for $k = 4l+1$,

$$\begin{bmatrix} 1 \\ 1/k \end{bmatrix} A_k^s = \begin{bmatrix} 1 \\ (\frac{1}{2}(k+1))^s/k \end{bmatrix}$$

and for $k = 4l+3$,

$$\begin{bmatrix} 1 \\ 1/k \end{bmatrix} A_k^s = \begin{bmatrix} 1 \\ (\frac{1}{2}(k-1))^s/k \end{bmatrix}$$

for every positive integer s. It therefore follows that

$$\begin{bmatrix} 1 \\ 1/k \end{bmatrix} A_k^l = \begin{bmatrix} 1 \\ 1/k \end{bmatrix}$$

only when $(\frac{k+1}{2})^s$ is congruent to ± 1 mod $2k$ for $k = 4l+1$ and when $(\frac{k-1}{2})^s$ is congruent to ± 1 mod $2k$ for $k = 4l+3$.

We identify the puncture determined by ∞ with the characteristic class of $\begin{bmatrix} 1 \\ 1/k \end{bmatrix}$. Now consider the orbit of $\begin{bmatrix} 1 \\ 1/k \end{bmatrix}$ under the cyclic group $< A_k >$. If the orbit consists of $(k-1)/2$ elements then we have obtained the first row of the list of characteristic classes. Under the same circumstances, the orbit under $< A_k >$ of the puncture determined by ∞ is the first row of the list of punctures. There is thus an obvious pairing between the first rows of the two lists. Since the generators of Γ provide us a with way of obtaining the subsequent rows from the first one, it is clear that we may obtain a bijection between the two lists. Unfortunately, while it is true that the orbit of $\begin{bmatrix} 1 \\ 1/k \end{bmatrix}$ under $< A_k >$ can not contain more than $(k-1)/2$ elements, it can contain fewer elements. The first occurrence of this phenomenon is for $k = 17$. We will not pursue this interesting number theoretic question here but will observe that it really does not affect our correspondence between punctures and characteristics. In fact, the above is a very nice way of obtaining the correspondence; but if all we wish to do is establish a correspondence, we can proceed in a different way exploiting the fact that each puncture on $\mathbb{H}^2/\Gamma(k)$ is the projection to this surface of an image of ∞ under some element of Γ. We will need some preliminaries which are developed in the next section.

5. The homomorphism from $\Gamma/\Gamma(k)$ to $\mathcal{S}_{n(k)}$

5.1 INVARIANT CHARACTERISTIC CLASSES. In §3, we have shown that the action of the unimodular matrices on characteristics defines a group action of Γ on the characteristic classes. Fix a prime k. We now show that the action defined gives rise to a representation of $\Gamma/\Gamma(k)$ as a subgroup of $\mathcal{S}_{n(k)}$, the permutation group on $n(k)$ elements. For this purpose, for $k > 2$, let us denote by $X(k)$ the finite set of $(k^2-1)/2$ characteristics discussed in the beginning of §3 (thus $X(k)$ consists of classes represented by characteristics $\begin{bmatrix} m/k \\ m'/k \end{bmatrix}$ with m and m' odd integers (not both equal to a multiple of k). We let $X(2)$ denote the three even integral theta characteristics.

LEMMA 1: *The set $X(k)$ is invariant under the action of Γ on characteristic classes.*

Proof: The proof for $k = 2$ is well known and straight forward; hence left to the reader. So assume that k is an odd prime. The image of $\begin{bmatrix} m/k \\ m'/k \end{bmatrix}$ under the action of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ is $\begin{bmatrix} \frac{(am+cm')}{k} - ac \\ \frac{(bm+dm')}{k} + bd \end{bmatrix}$. This characteristic (it obviously is not the class of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$) represents a class in $X(k)$ as long as not both a and c are even and not both b and d are even (the excluded cases are impossible since they would contradict the fact that $ad - bc = 1$). ∎

Proposition 1 and the above lemma now allow us to define a homomorphism η of Γ into $\mathcal{S}_{n(k)}$ by sending the motion $\gamma \in \Gamma$ to the permutation of the set

$X(k)$ that it induces. For $\gamma \in \Gamma$, the permutation $\eta(\gamma)$ sends the class of the characteristic χ to the class of the characteristic $\chi\gamma$.

We show now that the kernel of this homomorphism is $\Gamma(k)$. Again, we treat only the case $k > 2$. Consider an arbitrary element of our space of characteristics

$$\chi = \begin{bmatrix} m/k \\ m'/k \end{bmatrix}$$

with m and m' odd. The image of this characteristic under the motion

$$\begin{bmatrix} kr+1 & ks \\ kt & ku+1 \end{bmatrix} \in \Gamma(k)$$

is the characteristic

$$\begin{bmatrix} m/k + mr + m't - kt(kr+1) \\ m'/k + ms + m'u + ks(ku+1) \end{bmatrix}.$$

This characteristic is equivalent to χ provided $mr+m't-kt(kr+1)$ and $ms+m'u+ks(ku+1)$ are even integers. This fails to happen only if r is odd and t is even or u is odd and s is even in which cases we contradict that $(kr+1)(ku+1) - k^2st = 1$.

The above argument shows that $\Gamma(k)$ is contained in the kernel of the homomorphism η. We show next that if $\gamma \in \Gamma$ is not an element of $\Gamma(k)$, then there is a characteristic which is not fixed by it. As usual, let

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and assume that b is not congruent to zero mod k. Then we see that the (class of the) characteristic $\begin{bmatrix} 1/k \\ 1 \end{bmatrix}$ is not fixed by γ. Similarly, if c is not congruent to zero mod k, then the characteristic $\begin{bmatrix} 1 \\ 1/k \end{bmatrix}$ is not fixed by γ. It thus follows that we may assume that b and c are congruent to zero mod k. It follows that ad is congruent to one mod k. Consider now the characteristic $\begin{bmatrix} 1/k \\ 1/k \end{bmatrix}$. It is mapped by γ to $\begin{bmatrix} \frac{a}{k}+m \\ \frac{d}{k}+n \end{bmatrix}$ (for some integers m and n) which can be equivalent to its preimage only when a and d are both congruent to either ± 1 mod k. This means that γ is in $\Gamma(k)$.

We thus have a well defined monomorphism

$$\eta\colon \Gamma/\Gamma(k) \to \mathcal{S}_{n(k)}.$$

Remark 2: If, as usual, we represent elements of $\Gamma/\Gamma(k)$ by Möbius transformations (read from right to left) and elements of $\mathcal{S}_{n(k)}$ as permutations (read from left to right) of the first $n(k)$ natural numbers, then η is an antihomomrphism since for all γ_1 and γ_2 in Γ

$$\eta(\gamma_1 \circ \gamma_2) = \eta(\gamma_2)\eta(\gamma_1). \qquad \blacksquare$$

LEMMA 2: *Let*

$$\chi = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2$$

represent a characteristic class. Then χ is fixed by Γ_∞ if and only if ϵ is an odd integer and so without loss of generality we may assume $\epsilon = 1$ (hence $0 \le \epsilon' \le 1$ is arbitrary) or ϵ is an even integer which we may take as $\epsilon = 0$ and in this case $\epsilon' = \frac{1}{2}$.

Proof: The group Γ_∞ is generated by the motion B: $z \mapsto z + 1$. Since

$$\chi B = \begin{bmatrix} \epsilon \\ \epsilon + \epsilon' + 1 \end{bmatrix},$$

the if part of the lemma is trivial. For the converse, without loss of generality $0 \le \epsilon \le 1$ and $0 \le \epsilon' \le 2$. Assume first that $\epsilon = 0$. Thus we may assume $0 \le \epsilon' \le 1$. We are assuming that the characteristic $\begin{bmatrix} 0 \\ \epsilon' \end{bmatrix}$ is equivalent to the characteristic $\begin{bmatrix} 0 \\ \epsilon'+1 \end{bmatrix}$. Since $0 \le \epsilon' \le 1$, the only possibility is that $\epsilon' = \frac{1}{2}$. Assume next that $0 < \epsilon \le 1$. We conclude in this case that $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is equivalent to $\begin{bmatrix} \epsilon \\ \epsilon+\epsilon'+1 \end{bmatrix}$. From which it follows that $\epsilon + 1 = 2$. $\blacksquare$

5.2 THE CORRESPONDENCE BETWEEN PUNCTURES AND CHARACTERISTIC CLASSES (CONCLUSION). As before, we identify the puncture determined by ∞ with the characteristic class of

$$\chi_o = \begin{bmatrix} 1 \\ \frac{1}{k} \end{bmatrix}.$$

For each $\gamma \in \Gamma$, we identify the puncture corresponding to $\gamma^{-1}(\infty)$ with the characteristic class of $\begin{bmatrix} 1 \\ 1/k \end{bmatrix}\gamma$. This is a characteristic class in $X(k)$ as a result of Lemma 1. We need to show that our correspondence is well defined and injective. These facts follow from our claim that for all elements γ_1 and γ_2 of Γ,

$$\chi_o\gamma_1 \equiv \chi_o\gamma_2 \quad \text{if and only if } \gamma_1^{-1}(\infty) = \gamma_2^{-1}(\infty) \bmod \Gamma(k).$$

This last claim is reduced to showing that for $\gamma \in \Gamma$,

$$\chi_o\gamma \equiv \chi_o \quad \text{if and only if } \gamma^{-1}(\infty) = \infty \bmod \Gamma(k).$$

Let $\gamma \in \Gamma$ fix the characteristic class of χ_o. Write, as usual,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From

$$\begin{bmatrix} 1 \\ \frac{1}{k} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + \frac{c}{k} - ac \\ b + \frac{d}{k} + bd \end{bmatrix} \equiv \begin{bmatrix} 1 \\ \frac{1}{k} \end{bmatrix}$$

we see that c is congruent to 0 mod k and d is congruent to ± 1 mod k. If $c = 0$, then γ^{-1} fixes ∞. If $c \neq 0$, then $\gamma^{-1}(\infty) = -d/c$ is not equal but is $\Gamma(k)$-equivalent to ∞. The converse is easily established; that is, if $\gamma^{-1}(\infty)$ is $\Gamma(k)$-equivalent to ∞, then the theta characteristics χ_0 and $\chi_o\gamma$ are equivalent. Since $X(k)$ and the set of punctures on $\mathbb{H}^2/\Gamma(k)$ have the same cardinality $((k^2-1)/2)$, our correspondence is also surjective.

5.3 The action of Γ on classes and punctures. Let $\mathcal{I}$ denote the map from characteristic classes to punctures defined above. If $x \in \mathbb{R} \cup \infty$ is a cusp for $\Gamma(k)$ and $\gamma \in \Gamma$, then $\gamma(x)$ is another cusp for $\Gamma(k)$. This defines the **left permutation** of the punctures of $\mathbb{H}^2/\Gamma(k)$. Similarly, the **right permutation** of the punctures is defined by sending x to $\gamma^{-1}(x)$. In terms of these group actions the map $\mathcal{I}$ is defined by the relation (with $\gamma \in \Gamma$)

$$\mathcal{I}(\chi_o\gamma) = \gamma^{-1}(\infty) = \gamma^{-1}(\mathcal{I}(\chi_o)).$$

THEOREM 1: *The image in $S_{n(k)}$ of Γ under the homomorphism η is isomorphic to the group of conformal self maps of $\mathbb{H}^2/\Gamma(k)$. The action of Γ on the characteristic classes $X(k)$ corresponds to the right permutation of the punctures; that is, for each characteristic class $\chi \in X(k)$ and all $\gamma \in \Gamma$, we have*

$$\mathcal{I}(\chi\gamma) = \gamma^{-1}(\mathcal{I}(\chi)).$$

Proof: The image of the homomorphism η is isomorphic to $\Gamma/\Gamma(k)$ which is well known to be the group of conformal self maps of the Riemann surface $\mathbb{H}^2/\Gamma(k)$. The remark concerning the permutation of the punctures is a consequence of our identification of the characteristics with the punctures via the map $\mathcal{I}$; specifically, the characteristic χ is given as $\chi_o\gamma_o$ for some $\gamma_o \in \Gamma$. Hence

$$\begin{aligned}\mathcal{I}(\chi\gamma) &= \mathcal{I}((\chi_o\gamma_o)\gamma) = (\gamma_o \circ \gamma)^{-1}(\mathcal{I}(\chi_o)) \\ &= \gamma^{-1}(\gamma_o^{-1}(\mathcal{I}(\chi))) = \gamma^{-1}(\mathcal{I}(\chi\gamma_o)) = \gamma^{-1}(\mathcal{I}(\chi)).\end{aligned} \quad \blacksquare$$

6. Strange combinatorics

We have encountered in our work two interesting sets $\mathcal{P}_1$ and $\mathcal{Q}_1$ (we will redefine them below) that are in one to one (almost canonical) correspondence. We fix an odd prime k. The set $\mathcal{Q}_1$ is the quotient of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with x and y relatively prime integers by the equivalence relation $\begin{bmatrix} x \\ y \end{bmatrix}$ is equivalent to $\begin{bmatrix} x' \\ y' \end{bmatrix}$ if and only if

$$\pm \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} x' \\ y' \end{bmatrix} \bmod k.$$

We define the set $\mathcal{P}_1'$ as the quotient of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with x and y odd integers by the equivalence relation $\begin{bmatrix} x \\ y \end{bmatrix}$ is equivalent to $\begin{bmatrix} x' \\ y' \end{bmatrix}$ if and only if

$$\pm \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} x' \\ y' \end{bmatrix} \bmod 2k.$$

The set $\mathcal{P}_1$ is obtained by deleting from $\mathcal{P}_1'$ the equivalence class of $\begin{bmatrix} k \\ k \end{bmatrix}$. We see no direct (combinatorial) reason for the two sets to have the same cardinality.

7. Theta constants

7.1 Preliminaries. In this section we define the theta functions and theta constants and list some of the properties we shall be using. For the convenience of the reader we shall in some cases provide proofs of results even though they are not new. General references for this section are Chapters I and II of [4] and Chapter VI of [1]. We begin with a

Definition 2: The **theta function with characteristic** $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ is defined by the following series which converges uniformly and absolutely on compact subsets of $\mathbb{C} \times \mathbb{H}^2$:

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi\imath \left\{ (1/2)\left(n+\frac{\epsilon}{2}\right)^2 \tau + \left(n+\frac{\epsilon}{2}\right)\left(z+\frac{\epsilon'}{2}\right) \right\}. \quad \blacksquare$$

The theta functions satisfy the following properties: For m, n integers,

$$(1)\quad \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z+n+m\tau,\tau)=\exp 2\pi\imath\left\{\frac{n\epsilon-m\epsilon'}{2}-mz-\frac{m^2}{2}\tau\right\}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau).$$

More generally, for m, n arbitrary real numbers

$$(2)\quad \begin{aligned}&\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z+\tau\frac{m}{2}+\frac{n}{2},\tau)\\ &=\exp 2\pi\imath\left\{-\frac{1}{2}mz-\frac{1}{8}m^2\tau-\frac{1}{4}m(\epsilon'+n)\right\}\theta\begin{bmatrix}\epsilon+m\\ \epsilon'+n\end{bmatrix}(z,\tau).\end{aligned}$$

In fact, a double application of this formula with n and m integers gives the previous formula because of the next identity. For m and n integers,

$$(3)\quad \theta\begin{bmatrix}\epsilon+2m\\ \epsilon'+2n\end{bmatrix}(z,\tau)=\exp\pi\imath\{\epsilon n\}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau).$$

Further,

$$(4)\quad \theta\begin{bmatrix}-\epsilon\\ -\epsilon'\end{bmatrix}(z,\tau)=\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-z,\tau).$$

All of the above properties follow immediately from the definition of the theta function and the fact that the defining series is absolutely convergent. These properties are the manifestation of the fact that there really is only one theta function

$$\theta\begin{bmatrix}0\\ 0\end{bmatrix}=\theta,$$

and the theta functions with characteristic are expressed in terms of this function by the formula

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)=\exp 2\pi\imath\left\{\frac{1}{8}\epsilon^2\tau+\frac{1}{2}\epsilon z+\frac{1}{4}\epsilon\epsilon'\right\}\theta\left(z+\frac{\epsilon'}{2}+\tau\frac{\epsilon}{2},\tau\right).$$

The **theta constants** are defined by setting the variable $z=0$ in the definition of the theta functions. We shall abbreviate

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$$

when there can be no confusion. Formulae (3) and (4) show that the theta constants raised to the $2k$ power depends only on the class of the theta characteristic $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ provided ϵ is rational of the form m/k with $m\in\mathbb{Z}$.

7.2 THE TRANSFORMATION FORMULA. A property of theta functions deeper than the previous identities and which is central to the development of this paper is contained in the next formula. For any characteristic

$$\chi = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2,$$

and any element

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $\mathrm{SL}(2, \mathbb{Z})$, we have

$$(5) \qquad \frac{\exp \pi i \left\{ \frac{-cz^2}{c\tau+d} \right\} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right)}{\theta \begin{bmatrix} a\epsilon + c\epsilon' - ac \\ b\epsilon + d\epsilon' + bd \end{bmatrix} (z, \tau)} = \kappa \left(\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}, \gamma \right) (c\tau + d)^{1/2},$$

for all $z \in \mathbb{C}$, $\tau \in \mathbb{H}^2$, where $\kappa \left(\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right], \gamma \right)$ is a constant depending on the characteristic $\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix} \right]$ and the matrix γ. We will show that

$$(6) \qquad \begin{aligned} \kappa \left(\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}, \gamma \right) = \pm \exp 2\pi i \{ &- \frac{1}{4}(a\epsilon + c\epsilon')bd \\ &- \frac{1}{8}(ab\epsilon^2 + cd\epsilon'^2 + 2bc\epsilon\epsilon') \} \kappa \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \gamma \right), \end{aligned}$$

where the choice of sign depends both on the characteristic χ and the matrix γ.

THEOREM 2: *Let*

$$\chi = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2 \quad \text{and} \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

The theta constants satisfy the transformation rule

$$(7) \qquad \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \gamma(\tau)) = \kappa (c\tau + d)^{1/2} \theta \begin{bmatrix} a\epsilon + c\epsilon' - ac \\ b\epsilon + d\epsilon' + bd \end{bmatrix} (0, \tau),$$

for all $\tau \in \mathbb{H}^2$, where $\kappa = \kappa(\chi, \gamma)$ is the constant depending on the characteristic χ and the matrix γ defined by the above formula (6).

Proof: We begin by establishing (5). Fix τ and let g be the function defined by the left side of the equality. It is obviously a meromorphic function of z on

C. Our first task is to show that g is a doubly periodic function for the lattice generated by 1 and τ. Towards this end we begin with an examination of the periodicity of the function

$$z \mapsto \theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right).$$

We write

$$\frac{1}{c\tau+d}=a-c\frac{a\tau+b}{c\tau+d} \tag{8}$$

and

$$\frac{\tau}{c\tau+d}=-b+d\frac{a\tau+b}{c\tau+d}, \tag{9}$$

and conclude from these trivial identities that

$$\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z+1}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)$$

$$=\exp 2\pi\imath\left\{\frac{a\epsilon+c\epsilon'}{2}+c\frac{z}{c\tau+d}-\frac{c^2}{2}\frac{a\tau+b}{c\tau+d}\right\}\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)$$

and

$$\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z+\tau}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)$$

$$=\exp 2\pi\imath\left\{\frac{-b\epsilon-d\epsilon'}{2}-d\frac{z}{c\tau+d}-\frac{d^2}{2}\frac{a\tau+b}{c\tau+d}\right\}\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right).$$

It is immediate that

$$\exp\pi\imath\left\{\frac{-c(z+1)^2}{c\tau+d}\right\}=\exp\pi\imath\left\{\frac{-2cz}{c\tau+d}-\frac{c}{c\tau+d}\right\}\exp\pi\imath\left\{\frac{-cz^2}{c\tau+d}\right\}$$

and

$$\exp\pi\imath\left\{\frac{-c(z+\tau)^2}{c\tau+d}\right\}=\exp\pi\imath\left\{\frac{-2c\tau z}{c\tau+d}-\frac{c\tau^2}{c\tau+d}\right\}\exp\pi\imath\left\{\frac{-cz^2}{c\tau+d}\right\}.$$

Finally

$$\theta\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](z+1,\tau)=\exp\pi\imath\{a\epsilon+c\epsilon'-ac\}\theta\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](z,\tau)$$

and

$$\theta\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](z+\tau,\tau)$$

$$= \exp 2\pi\imath\left\{-z-\frac{\tau}{2}-\frac{b\epsilon+d\epsilon'+bd}{2}\right\}\theta\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](z,\tau).$$

It is easy to see that we have arranged our functions to have cancellation; that is,

$$g(z+1)=g(z)=g(z+\tau), \qquad \text{all } z\in\mathbb{C}.$$

The meromorphic function g has (at most) a single simple pole at a single point (in a period parallelogram). Hence it is a constant which depends on τ, the characteristic $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ and the matrix γ. We fix for the moment the characteristic and matrix, and denote the constant (as a function of τ) by $\kappa(\tau)$. To obtain the formula for $\kappa(\tau)$, we rewrite our basic identity as

$$\exp\pi\imath\left\{\frac{-cz^2}{c\tau+d}\right\}\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=\kappa(\tau)\theta\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](z,\tau)$$

and expand both sides in a power series (in z) about the origin. We begin with

$$\exp\pi\imath\left\{\frac{-cz^2}{c\tau+d}\right\}=1-\frac{\pi\imath cz^2}{c\tau+d}+\cdots,$$

$$\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)$$

$$+\frac{\theta'\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\frac{a\tau+b}{c\tau+d})}{c\tau+d}z+\frac{\theta''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\frac{a\tau+b}{c\tau+d})}{2(c\tau+d)^2}z^2+\frac{\theta'''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\frac{a\tau+b}{c\tau+d})}{6(c\tau+d)^3}z^3+\cdots,$$

and

$$\theta\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](z,\tau)=\theta\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](0,\tau)+\theta'\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](0,\tau)z$$

$$+\frac{1}{2}\theta''\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](0,\tau)z^2+\frac{1}{6}\theta'''\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](0,\tau)z^3+\cdots,$$

where the prime (′) after the theta function denotes differentiation with respect to z. Equating coefficients of powers of z leads to (four equations that we need)

$$\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)=\kappa(\tau)\theta\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](0,\tau),$$

$$\frac{1}{c\tau+d}\theta'\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)=\kappa(\tau)\theta'\left[\begin{array}{c} a\epsilon+c\epsilon'-ac \\ b\epsilon+d\epsilon'+bd \end{array}\right](0,\tau),$$

$$\frac{1}{2(c\tau+d)^2}\theta''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)-\frac{\pi\imath c}{c\tau+d}\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)$$
$$=\frac{\kappa(\tau)}{2}\theta''\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau),$$

and

$$\frac{1}{6(c\tau+d)^3}\theta'''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)-\frac{\pi\imath c}{(c\tau+d)^2}\theta'\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right]\left(0,\frac{a\tau+b}{c\tau+d}\right)$$
$$=\frac{\kappa(\tau)}{6}\theta'''\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau).$$

Dividing the third of the above equations by the first, we obtain

$$\frac{\theta''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\gamma(\tau))}{\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\gamma(\tau))}\gamma'(\tau)=\frac{2\pi\imath}{c\tau+d}+\frac{\theta''\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau)}{\theta\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau)}. \tag{10}$$

Similarly, from the other two equations

$$\frac{\theta'''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\gamma(\tau))}{\theta'\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\gamma(\tau))}\gamma'(\tau)=\frac{6\pi\imath}{c\tau+d}+\frac{\theta'''\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau)}{\theta'\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau)}.$$

We now make use of the heat equation

$$\frac{\partial^2\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\tau)}{\partial z^2}=4\pi\imath\frac{\partial\theta\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\tau)}{\partial\tau},$$

which is easily established using the series expansion of the θ-function involved.

Differentiating the first of our four equations with respect to τ and using the heat equation yields

$$\frac{1}{4\pi\imath}\frac{\theta''\left[\begin{array}{c}\epsilon\\ \epsilon'\end{array}\right](0,\gamma(\tau))}{(c\tau+d)^2}$$
$$=\frac{\kappa(\tau)}{4\pi\imath}\theta''\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau)+\kappa'(\tau)\theta\left[\begin{array}{c}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{array}\right](0,\tau)$$

(the prime (′) denotes differentiation with respect to z for the θ-function (as before) and with respect to τ for κ). "Dividing by the first equation" results in

$$\frac{1}{4\pi\imath}\frac{1}{(c\tau+d)^2}\frac{\theta''\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\gamma(\tau))}{\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\gamma(\tau))}=\frac{1}{4\pi\imath}\frac{\theta''\begin{bmatrix}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{bmatrix}(0,\tau)}+\frac{\kappa(\tau)}{\kappa'(\tau)}.$$

Combining this last equation with (10) leads to the ordinary differential equation

$$\frac{\kappa(\tau)}{\kappa'(\tau)}=\frac{c}{2(c\tau+d)}$$

satisfied by our unknown function κ. From which it easily follows that

$$\kappa(\tau)=\kappa_o(c\tau+d)^{1/2},$$

where the constant κ_o depends on the characteristic $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ and the matrix γ.

It is a consequence of what we have already proved that

$$\exp\pi\imath\left\{\frac{-cz^2}{c\tau+d}\right\}\theta\begin{bmatrix}0\\ 0\end{bmatrix}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=$$

$$\kappa\left(\begin{bmatrix}0\\ 0\end{bmatrix},\gamma\right)(c\tau+d)^{1/2}\theta\begin{bmatrix}-ac\\ bd\end{bmatrix}(z,\tau).$$

Set

$$z=\zeta+\tau\frac{a\epsilon+c\epsilon'}{2}+\frac{b\epsilon+d\epsilon'}{2}$$

with ϵ,ϵ' arbitrary real numbers and obtain

$$\begin{aligned}&\exp\pi\imath\left\{\frac{-c(\zeta+\tau\frac{a\epsilon+c\epsilon'}{2}+\frac{b\epsilon+d\epsilon'}{2})^2}{c\tau+d}\right\}\theta\begin{bmatrix}0\\ 0\end{bmatrix}\left(\frac{\zeta+\tau\frac{a\epsilon+c\epsilon'}{2}+\frac{b\epsilon+d\epsilon'}{2}}{c\tau+d},\gamma(\tau)\right)\\ (11)\quad&=\kappa\left(\begin{bmatrix}0\\ 0\end{bmatrix},\gamma\right)(c\tau+d)^{1/2}\theta\begin{bmatrix}-ac\\ bd\end{bmatrix}\left(\zeta+\tau\frac{a\epsilon+c\epsilon'}{2}+\frac{b\epsilon+d\epsilon'}{2},\tau\right).\end{aligned}$$

We now note that from (2) it follows that

$$\begin{aligned}&\theta\begin{bmatrix}-ac\\ bd\end{bmatrix}\left(\zeta+\tau\frac{a\epsilon+c\epsilon'}{2}+\frac{b\epsilon+d\epsilon'}{2},\tau\right)\\ (12)\quad&=\exp 2\pi\imath\left\{-\frac{1}{2}(a\epsilon+c\epsilon')\zeta-\frac{1}{8}(a\epsilon+c\epsilon')^2\tau-\frac{1}{4}(a\epsilon+c\epsilon')(bd+b\epsilon+d\epsilon')\right\}\\ &\times\theta\begin{bmatrix}a\epsilon+c\epsilon'-ac\\ b\epsilon+d\epsilon'+bd\end{bmatrix}(\zeta,\tau).\end{aligned}$$

From (2), using (8) and (9), we see that

$$(13)\qquad \theta\begin{bmatrix}0\\0\end{bmatrix}\left(\frac{\zeta+\tau\frac{a\epsilon+c\epsilon'}{2}+\frac{b\epsilon+d\epsilon'}{2}}{c\tau+d},\gamma(\tau)\right)=$$

$$(14)\qquad \exp 2\pi\imath\left\{-\frac{1}{2}\epsilon\frac{\zeta}{c\tau+d}-\frac{1}{8}\epsilon^2\frac{a\tau+b}{c\tau+d}-\frac{1}{4}\epsilon\epsilon'\right\}\theta\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix}\left(\frac{\zeta}{c\tau+d},\tau\right).$$

From (11), using (12) and (14), we obtain a formula which has the form

$$\exp\pi\imath\left\{-\frac{c\zeta^2}{c\tau+d}\right\}\theta\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix}\left(\frac{\zeta}{c\tau+d},\gamma(\tau)\right)=$$

$$\kappa\left(\begin{bmatrix}0\\0\end{bmatrix},\gamma\right)(c\tau+d)^{1/2}\theta\begin{bmatrix}a\epsilon+c\epsilon'-ac\\b\epsilon+d\epsilon'+bd\end{bmatrix}(\zeta,\tau)E,$$

with $E=E(\chi,\gamma)$ a very complicated exponential expresion involving all the variables (depending on the characteristic χ and the matrix γ) we have used. With a little bit of perseverance, however, one can simplify E to be

$$\exp 2\pi\imath\left\{-\frac{1}{4}(a\epsilon+c\epsilon')bd-\frac{1}{8}(ab\epsilon^2+cd\epsilon'^2+2bc\epsilon\epsilon')\right\}$$

from which we obtain (6), our almost final formula for the constant κ. There now only remains the problem of computing $\kappa\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],\gamma\right)$ for arbitrary $\gamma\in$ SL$(2,\mathbb{Z})$. We do this for our three favorite generators for the group of unimodular matrices: $-I, A$, and B. It is trivial to show that for all characteristics $\left[\begin{smallmatrix}\epsilon\\\epsilon'\end{smallmatrix}\right]$,

$$\kappa\left(\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix},-I\right)=\pm\imath.$$

It follows from the definition of the theta function that

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,B(\tau))=\sum_{n=-\infty}^{n=\infty}\exp 2\pi\imath\left\{\frac{1}{2}n^2(\tau+1)\right\}=\sum_{n=-\infty}^{n=\infty}\exp 2\pi\imath\left\{\frac{1}{2}n^2\tau+n/2\right\}$$

$$=\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)$$

(we have used in the penultimate of the above equalities the fact that for integers N, $\exp\pi\imath N$ depends only on the parity of N and hence we can replace $\exp\pi\imath n^2$ by $\exp\pi\imath n$), so that

$$\kappa\left(\begin{bmatrix}0\\0\end{bmatrix},B\right)=\pm 1.$$

We now treat the case of the motion A. Here we observe that $\tau = \imath$ is a fixed point of the motion and since κ is independent of the point τ we compute at that point. We have

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, A(\imath)) = \kappa\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, A\right)\sqrt{\imath}\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \imath).$$

Since $A(\imath) = \imath$, we get

$$\kappa\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, A\right) = \frac{1}{\sqrt{\imath}}.$$

After some remarks on factors of automorphy, we will be able to derive information about $\kappa\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \gamma\right)$ for arbitrary $\gamma \in \mathrm{SL}(2, \mathbb{Z})$, from the information we derived (the formula for this mysterious quantity for the generators $-I$, A, and B of $\mathrm{SL}(2, \mathbb{Z})$) which will complete the proof of the theorem. ∎

7.3 FACTORS OF AUTOMORPHY. Let G be a subgroup of $\mathrm{SL}(2, \mathbb{R})$. We are particularly interested in groups whose images in the Möbius group $\mathrm{PSL}(2, \mathbb{R})$ are Fuchsian. We restrict our attention to the action of the group G on $\mathbb{H}^2$. A **factor of automorphy** for G is a nonvanishing function

$$e: G \times \mathbb{H}^2 \to \mathbb{C}^*$$

with

$$e(g, \cdot): \mathbb{H}^2 \to \mathbb{C}^*$$

holomorphic for all $g \in G$ and

$$e(g_1 g_2, \tau) = e(g_1, g_2(\tau))e(g_2, \tau)$$

for all g_1, $g_2 \in G$ and all $\tau \in \mathbb{H}^2$. For

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the map

$$e(g, \tau) = (c\tau + d)$$

defines a factor of automorphy for $\mathrm{SL}(2, \mathbb{R})$. However $(c\tau + d)^{1/2}$ is not a factor of automorphy for (the smaller) group $\mathrm{SL}(2, \mathbb{Z})$ since it assigns either $\pm\sqrt{\imath}$, an 8-th root of unity, to the pair $\left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \imath\right)$. This is impossible since $\imath$ is a fixed point of the motion determined by the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ which has order 4.

In general, for fixed characteristic χ, powers of $\kappa(\chi,\cdot)$ will define factors of automorphy for subgroups of $\mathrm{SL}(2,\mathbb{Z})$ that are independent of τ. In order to derive multiplicative properties of the functions κ, we must use quantities that depend only on characteristic classes. We start with the case

$$\chi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We write our basic identity (with m and m' integers) as

$$\theta^2 \begin{bmatrix} m \\ m' \end{bmatrix} (0,\gamma(\tau)) = \kappa^2 \left(\begin{bmatrix} m \\ m' \end{bmatrix}, \gamma \right) (c\tau + d)\theta^2 \begin{bmatrix} am + cm' - ac \\ bm + dm' + bd \end{bmatrix} (0,\tau).$$

In the above equation we can replace the charactristics by their classes and we conclude that

$$\kappa^2 \left(\begin{bmatrix} m \\ m' \end{bmatrix}, \gamma_1\gamma_2 \right) = \kappa^2 \left(\begin{bmatrix} m \\ m' \end{bmatrix}, \gamma_1 \right) \kappa^2 \left(\begin{bmatrix} a_1 m + c_1 m' - a_1 c_1 \\ b_1 m + d_1 m' + b_1 d_1 \end{bmatrix}, \gamma_2 \right)$$

for all

$$\gamma_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad \text{and} \quad \gamma_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

in $\mathrm{SL}(2,\mathbb{Z})$ and all integers m, m'. We conclude from the above identity and (6) that

$$\begin{aligned} \kappa^2 \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \gamma_1\gamma_2 \right) = & \kappa^2 \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \gamma_1 \right) \kappa^2 \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \gamma_2 \right) \\ & \times \exp 2\pi\imath \Big\{ -\frac{1}{2}(-a_1c_1a_2 + b_1d_1c_2)b_2d_2 \\ & - \frac{1}{4}(a_1^2c_1^2a_2b_2 + b_1^2d_1^2c_2d_2 - 2a_1b_1c_1d_1b_2d_2) \Big\}. \end{aligned} \tag{15}$$

The above formula and the fact that $\kappa\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \gamma\right)$ is an 8-th root of unity for $\gamma = -I$, A, and B allow us to conclude (because $-I$, A, and B generate $\mathrm{SL}(2,\mathbb{Z})$) that $\kappa\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \gamma\right)$ is an 8-th root of unity for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Our next claim is that $\kappa\left(\left[\begin{smallmatrix} m/k \\ m'/k \end{smallmatrix}\right], \gamma\right)$ is an $8k$-th root of unity for every motion γ in the preimage in $\mathrm{SL}(2,\mathbb{Z})$ of $\Gamma(k)$, for all primes k and all integers m and m'. Also, for integral characteristics $\left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right]$, $\kappa\left(\left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right], \gamma\right)$ is an 8-th root of unity for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Both of these claims follow directly from (6) and (15).

Remark 3: We need to distinguish formulae that are valid for theta characteristics from those that only hold for characteristic classes. In this regard, we observe that (6) and (7) are identities for characteristics while (15) is true only on the level of characteristic classes (which sometimes makes it difficult to apply (7) on the level of theta characteristics). Many of the formulae involving motions $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ depend on the choice of matrix in $\mathrm{SL}(2,\mathbb{Z})$ that represents the motion. ∎

8. Automorphic forms

The theta constants (as functions of τ) yield automorphic forms (actually cusp forms) for the groups $\Gamma(k)$. For details about automorphic forms see Chapter III of [3]. For $q \in \mathbb{Z}$, $q \geq 2$, we let $\mathbf{A}_q(\mathbb{H}^2,\Gamma(k))$ denote the space of q-cusp forms (supported on $\mathbb{H}^2$) for the group $\Gamma(k)$. We restrict our attention to the case where k is an odd prime and $q = 2k$. It follows from Riemann–Roch that

$$\dim \mathbf{A}_{2k}(\mathbb{H}^2,\Gamma(k)) = (2q-1)(p-1) + (q-1)n = \frac{(k^2-1)(4k^2-k-6)}{24}.$$

For the characteristic

$$\chi = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix},$$

we define

$$\varphi\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} : \tau \mapsto \theta^{8k}\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(0,\tau).$$

THEOREM 3: *For a fixed prime $k > 2$, the complex valued function*

$$\varphi\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbf{A}_{2k}(\mathbb{H}^2,\Gamma(k)),$$

provided the characteristic is of the form $\begin{bmatrix} m/k \\ m'/k \end{bmatrix}$ with m and m' odd integers. This cusp form depends only on the class of the characteristic $\begin{bmatrix} m/k \\ m'/k \end{bmatrix}$.

Proof: The function $\varphi\begin{bmatrix} m/k \\ m'/k \end{bmatrix}$ is holomorphic and nonzero on $\mathbb{H}^2$. Since the theta constant is raised to the $8k$-th power (actually for this statement $2k$-th power suffices), the resulting function depends not on the theta characteristic, but only on its class. Since $\kappa(\chi,\gamma)$ is an $8k$-th root of unity for all $\gamma \in \Gamma(k)$*

* Technically we should say " for all γ in the preimage of $\Gamma(k)$ in $\mathrm{SL}(2,\mathbb{Z})$." This abuse of language should not cause any confusion.

and the characteristic class determined by $\left[\begin{smallmatrix} m/k \\ m'/k \end{smallmatrix}\right]$ is fixed by $\Gamma(k)$, $\varphi\left[\begin{smallmatrix} m/k \\ m'/k \end{smallmatrix}\right]$ is an automorphic form for the group $\Gamma(k)$. It remains to show that this function is a cusp form. We will do so by showing that $\varphi\left[\begin{smallmatrix} m/k \\ m'/k \end{smallmatrix}\right]$ extends to $\mathbb{H}^2 \cup \mathbb{Q} \cup \{\infty\}$ and examining the projection to $(\mathbb{H}^2 \cup \mathbb{Q} \cup \{\infty\})/\Gamma(k)$ of the divisor of this extension.

We compute

$$\begin{aligned}\theta\left[\begin{array}{c} \frac{m}{k}, \\ \frac{m'}{k} \end{array}\right](0,\tau) &= \sum_{n\in\mathbf{Z}} \exp 2\pi\imath\left\{(1/2)(n+\frac{m}{2k})^2\tau + (n+\frac{m}{2k})\frac{m'}{2k}\right\} \\ &= \sum_{n\in\mathbf{Z}} \exp\left\{\frac{2\pi\imath\tau}{k}(n+\frac{m}{2k})^2\frac{k}{2}\right\}\exp\left\{\pi\imath(n+\frac{m}{2k})\frac{m'}{k}\right\} \\ &= \sum_{n\in\mathbf{Z}} \zeta^{\frac{kn^2}{2}+\frac{nm}{2}+\frac{m^2}{8k}}\exp\left\{\pi\imath(n+\frac{m}{2k})\frac{m'}{k}\right\},\end{aligned}$$

where $\zeta = \exp\{2\pi\imath\tau/k\}$. Since both k and m are odd, it follows that $kn^2/2 + nm/2$ is an integer. Further for all $n \in \mathbb{Z}$, $kn^2/2 + nm/2$ is positive except for n in the closed interval $[-m/k, 0]$. We conclude that we may rewrite our sum as

$$\begin{aligned}&\theta\left[\begin{array}{c} \frac{m}{k}, \\ \frac{m'}{k} \end{array}\right](0,\tau) \\ &= \zeta^{\frac{m^2}{8k}}\left(\exp\left\{\frac{\pi\imath mm'}{2k^2}\right\} + \exp\left\{\frac{\pi\imath(m-2k)m'}{2k^2}\right\}\zeta^{\frac{k}{2}-\frac{m}{2}} + \sum_{n>\frac{k}{2}-\frac{m}{2}}^{\infty} c_n\zeta^n\right),\end{aligned}$$

for $1 \le m < k-1$ and

$$\begin{aligned}&\theta\left[\begin{array}{c} \frac{m}{k}, \\ \frac{m'}{k} \end{array}\right](0,\tau) \\ &= \zeta^{\frac{m^2}{8k}}\left(\exp\left\{\frac{\pi\imath(m-2k)m'}{2k^2}\right\}\zeta^{\frac{k}{2}-\frac{m}{2}} + \exp\left\{\frac{\pi\imath mm'}{2k^2}\right\} + \sum_{n\ge\frac{k}{2}+\frac{m}{2}}^{\infty} c_n\zeta^n\right),\end{aligned}$$

for $k \le m \le 2k-1$. In each case, the constants c_n are (in principle) computable. For $k = m$ the last sum simplifies to

$$\theta\left[\begin{array}{c} 1 \\ \frac{m'}{k} \end{array}\right](0,\tau) = \zeta^{k/8}\left(\exp\left\{\frac{-\pi\imath m'}{2k}\right\} + \exp\left\{\frac{\pi\imath m'}{2k}\right\} + \sum_{n\ge k}^{\infty} c_n\zeta^n\right).$$

The two constants in the last equation add to zero if and only if $k = m'$. In what follows we use the notion of reduced order. The definition of reduced order (for a meromorphic automorphic form) is to be found in Chapter III of [3]. We conclude that

$$\text{red ord}_\infty \varphi \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix} = m^2 - 2k,$$

if $1 \le m < k-1$ and

$$\text{red ord}_\infty \varphi \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix} = (m-2k)^2 - 2k,$$

if $k < m \le 2k-1$. For $k = m$, we have

$$\text{red ord}_\infty \varphi \begin{bmatrix} 1 \\ \frac{m'}{k} \end{bmatrix} = k(k-2)$$

(which is the limiting case of the two previous formulae) provided $m' \neq k$. (For $m' = k$, the theta constant (this is a function of τ!) vanishes identically.)

If φ is a meromorphic automorphic q-form for Γ defined on $\mathbf{H}^2$ and C is a Möbius transformation fixing $\mathbf{H}^2$, then

$$\psi = C_q^*(\varphi) = (\varphi \circ C)(C')^q \tag{16}$$

is a meromorphic automorphic q-form for $C^{-1}\Gamma C$ also defined on $\mathbf{H}^2$ and

$$\text{red ord}_z \psi = \text{red ord}_{C(z)} \varphi$$

for all $z \in \mathbf{H}^2 \cup \{\text{parabolic fixed points of } C^{-1}\Gamma C\}$. For the forms under consideration we have

$$\varphi \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = (C^{-1})^*_{2k} \varphi \begin{bmatrix} a\epsilon + c\epsilon' - ac \\ b\epsilon + d\epsilon' + bd \end{bmatrix}$$

(where $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$), and hence also

$$\text{red ord}_z \varphi \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = \text{red ord}_{C(z)}^{-1} \varphi \begin{bmatrix} a\epsilon + c\epsilon' - ac \\ b\epsilon + d\epsilon' + bd \end{bmatrix},$$

for all $z \in \mathbb{H}^2 \cup \mathbb{Q} \cup \{\infty\}$. Similar statements hold for the value and Fourier series expansions at $\imath\infty$ for the forms under consideration.

We are now ready to prove that

$$\varphi \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix} \in \mathbf{A}_{2k}(\mathbb{H}^2, \Gamma(k)).$$

Since the automorphic form $\varphi\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ is regular on $\mathbb{H}^2$, we need only examine its behaviour at the cusps. If $x \in \mathbb{R} \cup \{\infty\}$ is a parabolic fixed point of $\Gamma(k)$, there exists a $C \in \mathrm{SL}(2,\mathbb{Z})$ with $C^{-1}(x) = \infty$. It follows that

$$\text{red ord}_x\, \varphi \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix} = \text{red ord}_\infty\, \varphi \begin{bmatrix} \frac{m_1}{k}, \\ \frac{m_1'}{k} \end{bmatrix}$$

for some pair of odd integers m_1 and m_1'. (Similar transformation rules hold for the Fourier series expansions of the automorphic forms $\varphi\left[\begin{smallmatrix}\frac{m}{k},\\ \frac{m'}{k}\end{smallmatrix}\right]$.) It then also follows that

$$\text{red ord}_x\, \varphi \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix} \geq 1 - 2k$$

which shows that $\varphi\left[\begin{smallmatrix}\frac{m}{k}\\ \frac{m'}{k}\end{smallmatrix}\right]$ is a $2k$-cusp form. ∎

Remark 4: For mm' even, the function $\varphi\left[\begin{smallmatrix}\frac{m}{k},\\ \frac{m'}{k}\end{smallmatrix}\right]$ is not an automorphic form for $\Gamma(k)$, because the characteristic class determined by $\left[\begin{smallmatrix}\frac{m}{k},\\ \frac{m'}{k}\end{smallmatrix}\right]$ is not fixed by each element of the group. However it still makes sense to compute the Fourier series expansions (valid for $\tau \in \mathbb{H}^2$ with large imaginary part) of the various theta functions. We will need

$$\theta \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix}(0,\tau) = \zeta^{m^2/8k}\left(\exp\left\{\pi\imath\frac{mm'}{2k^2}\right\} + o(1)\right)$$

for $0 \leq m < k$,

$$\theta \begin{bmatrix} 1 \\ \frac{m'}{k} \end{bmatrix}(0,\tau) = \zeta^{k/8}\left(2\cos\left\{\pi\frac{m'}{2k}\right\} + o(1)\right),$$

and

$$\theta \begin{bmatrix} \frac{m}{k}, \\ \frac{m'}{k} \end{bmatrix}(0,\tau) = \zeta^{k/2-m/2+m^2/8k}\left(\exp\left\{\pi\imath\left(-1+\frac{m}{2k}\right)\frac{m'}{k}\right\} + o(1)\right)$$

for $k < m \leq 2k$. ∎

Remark 5: The case $k = 2$ is known to the experts. Arguments similar to the ones used above show that

$$\text{red ord}_\infty\, \varphi \begin{bmatrix} 0 \\ m' \end{bmatrix} = -4$$

for $m' = 0$ or 1, and

$$\text{red ord}_\infty \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

In this case for m and $m' = 0$ or 1 with $mm' = 0$, $\varphi\left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right]$ is a holomorphic 4-form (but not a cusp form) for $\Gamma(2)$. As a matter of fact, if we set $\zeta = \exp\{\pi\imath\tau\}$, then we have for $m' = 0$ or 1,

$$\theta \begin{bmatrix} 0 \\ m' \end{bmatrix} (0,\tau) = \sum_{n\in\mathbf{Z}} \zeta^{n^2} \exp\{\pi\imath m' n\} = 1 + (-1)^{m'} 2\zeta + \mathrm{O}(\zeta^4)$$

and

$$\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0,\tau) = \zeta^{1/4} \sum_{n\in\mathbf{Z}} \zeta^{n^2+n} = 2\zeta^{1/4}(1 + \zeta^2 + \mathrm{O}(\zeta^6)),$$

as well as the expansions given in the previous remark for fractional characteristics. ∎

9. $k = 2$

For the arbitrary positive prime k, we consider the projection

$$P \colon \mathbb{H}^2 \to \mathbb{H}^2/\Gamma(k)$$

and its extension to $\mathbb{Q} \cup \{\infty\}$. The image of this extended map is closed. For $x \in \mathbb{H}^2 \cup \mathbb{Q} \cup \{\infty\}$, we will often write P_x for $P(x)$.

The case $k = 2$ differs from the other cases because 2 is the (only) even prime. Using the definition of characteristic classes given in section 3, we now consider the characteristic classes represented by vectors of the form $\left[\begin{smallmatrix} m/2 \\ m'/2 \end{smallmatrix}\right]$ with m and m' even integers. This set consists of the four classical integer characteristics. If we delete from this list the characteristic $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, we have the set of three even characteristics. If we now consider as well all vectors of the form $\left[\begin{smallmatrix} m/2 \\ m'/2 \end{smallmatrix}\right]$ with m and m' arbitrary integers, we add six additional classes to our list. This was already described in the introduction. We already explained there how we can obtain a homomorphism from Γ onto S_3. A similar argument to the one given in section 6 which we shall not repeat shows that the kernel of the homomorphism is precisely $\Gamma(2)$. The group $\Gamma(2)$ acts nontrivially on the six additional classes and as we have described in the introduction, gives us a homomorphism of $\Gamma(2)$ onto the Klein 4-group. It is easy to check that the kernel of this homomorphism is

$\Gamma(4)$, so that $\Gamma(2)/\Gamma(4)$ is just the Klein 4-group. More importantly for us though is the fact that the action of $\Gamma(2)$ on the six fractional classes has precisely three distinct orbits and each one can be associated in a natural way with one of the three even integral characteristics. The "natural way" is via the action of Γ, which also acts on the orbits. This gives us the correspondence mentioned in the introduction.

THEOREM 4: *We have the following proportionalities among theta constants:*

$$\frac{\theta^8\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\theta^8\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}}{\theta^4\begin{bmatrix}1\\ 0\end{bmatrix}}=\frac{\theta^8\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}\theta^8\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}}{\theta^4\begin{bmatrix}0\\ 1\end{bmatrix}}=\frac{\theta^8\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\theta^8\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\end{bmatrix}}{\theta^4\begin{bmatrix}0\\ 0\end{bmatrix}}.$$

The proof of this theorem requires some well known results concerning theta functions which we shall state without proof. Before we actually do this however, let us observe that what we have here is a statement about modular forms for the group $\Gamma(2)$. The statement is that each of the quotients is a modular 3-form for $\Gamma(2)$ and that the three quotients define the same form.

The main tool in the proof of the above theorem is the well known addition formula for theta constants and some easily proven facts about them. The addition formula (for theta constants) we state is actually true also for theta functions as well as in the higher genus case.

LEMMA 3: *For all characteristics* $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\delta\\ \delta'\end{smallmatrix}\right]$ *and all* $\tau\in\mathbb{H}^2$, *we have*

$$\begin{aligned}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)\theta\begin{bmatrix}\delta\\ \delta'\end{bmatrix}(0,\tau)=\theta&\begin{bmatrix}\frac{\epsilon+\delta}{2}\\ \epsilon'+\delta'\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}\frac{\epsilon-\delta}{2}\\ \epsilon'-\delta'\end{bmatrix}(0,2\tau)\\ &+\theta\begin{bmatrix}\frac{\epsilon+\delta}{2}-1\\ \epsilon'+\delta'\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}\frac{\epsilon-\delta}{2}+1\\ \epsilon'-\delta'\end{bmatrix}(0,2\tau).\end{aligned}$$

In particular, we have

$$\begin{aligned}\theta^2\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)=\theta&\begin{bmatrix}\epsilon\\ 2\epsilon'\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)\\ &+\theta\begin{bmatrix}\epsilon-1\\ 2\epsilon'\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau).\end{aligned}$$

Proof: See the appendix to Chapter II of [4]. ∎

As a consequence of the above addition formula, (3), and (4) one easily obtains the following set of equations

$$\theta^2\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0\\ 1\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau),$$

$$\theta^2\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0\\ 1\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau),$$

$$\theta^2\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0,2\tau)(\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)+\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau)),$$

$$\theta^2\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}\frac{1}{2}\\ 2\end{bmatrix}(0,2\tau)(\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)-\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau)),$$

$$\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}(0,2\tau)(\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)-i\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau)),$$

and

$$\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}\frac{1}{2}\\ 3\end{bmatrix}(0,2\tau)(\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)+i\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau)).$$

Another application of (3) and some agebraic manipulation yield

$$\theta^4\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0,\tau)\theta^4\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}(0,\tau)=\theta^4\begin{bmatrix}0\\ 1\end{bmatrix}(0,2\tau)\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)\theta^2\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau),$$

$$\begin{aligned}&\theta^4\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0,\tau)\theta^4\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}(0,\tau)\\ &=-\theta^4\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0,2\tau)(\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau))^2,\end{aligned}$$

and

$$\begin{aligned}&\theta^4\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0,\tau)\theta^4\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\end{bmatrix}(0,\tau)\\ &=-\theta^4\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}(0,2\tau)(\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1\\ 0\end{bmatrix}(0,2\tau))^2.\end{aligned}$$

It is important to observe that the right hand sides of the last three equations equal respectively to

$$\frac{1}{4}\theta^4\begin{bmatrix}0\\ 1\end{bmatrix}(0,2\tau)\theta^4\begin{bmatrix}1\\ 0\end{bmatrix}(0,\tau),$$

$$-\theta^4\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(0,2\tau)\theta^4\begin{bmatrix}0\\1\end{bmatrix}(0,\tau),$$

and

$$-\theta^4\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}(0,2\tau)\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,\tau).$$

Our final formulae are a consequence of the general addition formula noted above. They are:

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0\\1\end{bmatrix}(0,2\tau),$$

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)=2\theta^2\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(0,2\tau),$$

and

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)=-2i\theta^2\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}(0,2\tau).$$

Putting all this together yields the proportionality between the fourth powers of the classical theta constants (integer characteristic) and the products of theta constants with half integer characteristics. We have arrived at some identities among theta constants.

COROLLARY 1: *We have*

$$\theta^8\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\theta^8\begin{bmatrix}\frac{1}{2}\\\frac{3}{2}\end{bmatrix}=\theta^8\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\theta^8\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}+\theta^8\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}\theta^8\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}.$$

Proof: The proportionality and the classical identity among the fourth powers of the classical theta constants yields the above. ∎

Remark 6: The methods of the next section can yield alternate proofs of the proportionalities established in this section. ∎

10. $k = 3$

It is well known that $\mathbb{H}^2/\Gamma(3)$ is a four times punctured sphere; the punctures are the images of the parabolic fixed points $-1,\ 0,\ 1,\ \infty$ under the extension to the parabolic fixed points of the group $\Gamma(3)$ of the canonical projection $\mathbb{H}^2 \to \mathbb{H}^2/\Gamma(3)$; we denote these puncturess by $P_{-1},\ P_0,\ P_1$ and P_∞, respectively.

Recall that $\mathrm{SL}(2,\mathbb{Z})$ is the normalizer of $\Gamma(3)$ as well as equation (16). For $C \in \mathrm{SL}(2,\mathbb{Z})$, we have that ψ is a q-form for $\Gamma(3)$, whenever φ is. Taking

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

we see that

$$C_q^*\left(\varphi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\right) = \epsilon\varphi\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix},$$

with ϵ a root of unity, and hence

$$\operatorname{red}\operatorname{ord}_0 \varphi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \operatorname{red}\operatorname{ord}_\infty \varphi\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} = -5.$$

Similarly (using $C = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$), we see that

$$\operatorname{red}\operatorname{ord}_1 \varphi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \operatorname{red}\operatorname{ord}_\infty \varphi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = -5$$

and

$$\operatorname{red}\operatorname{ord}_{-1} \varphi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \operatorname{red}\operatorname{ord}_\infty \varphi\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} = 3;$$

from which it follows that the divisor $\left(\Phi\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}\right)$ of the projection $\Phi\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$ of $\varphi\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$ (to $\mathbb{H}^2/\Gamma(3)$) is given by

$$\left(\Phi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\right) = \frac{P_{-1}^3}{P_0^5 P_1^5 P_\infty^5}.$$

Similarly

$$\left(\Phi\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right) = \frac{P_0^3}{P_{-1}^5 P_1^5 P_\infty^5},$$

$$\left(\Phi\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}\right) = \frac{P_1^3}{P_{-1}^5 P_0^5 P_\infty^5}$$

and

$$\left(\Phi\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}\right) = \frac{P_\infty^3}{P_{-1}^5 P_0^5 P_1^5}.$$

It now makes sense to form the ratios

$$\frac{\varphi\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}{\varphi\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}} \quad \text{and} \quad \frac{\varphi\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}}{\varphi\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}}$$

and observe that these are automorphic functions for the group $\Gamma(3)$ whose projections to $\mathbb{H}^2/\Gamma(3)$ have divisors

$$\frac{P_{-1}^8}{P_0^8} \quad \text{and} \quad \frac{P_1^8}{P_0^8},$$

respectively. It follows that eighth roots (λ and g) of these functions provide covering maps of $\mathbb{H}^2/\Gamma(3)$ (for us there are obvious roots to choose; namely, ratios of cubes of theta constants). We have therefore proven the following

THEOREM 5: *The maps*

$$\tau \mapsto \frac{\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}}{\theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}} \quad \text{and} \quad \tau \mapsto \frac{\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}}{\theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}}$$

are holomorphic universal covering maps from $\mathbb{H}^2$ to the four times punctured sphere $\mathbb{H}^2/\Gamma(3)$.

Remark 7: Up to choice of signs, the ratios of the third powers of the theta constants that appear in the above theorem depend only on the characteristic classes (not the theta characteristics themselves). It is also easy to conclude directly from (7) and (6) that up to sign, the above ratios are automorphic functions on $\mathbb{H}^2$ for $\Gamma(k)$. The elimination of the ambiguities in signs (the fact that we have single rather than multivalued functions) follows from the nature of the singularities that our functions have and the simple connectivity of the compactification of $\mathbb{H}^2/\Gamma(3)$. The choice of the two characteristics is quite arbitrary. Any choice of two out of the four will give a covering map. ∎

Now any two covering maps that are automorphic with respect to the same group are post-related by a conformal map (Möbius transformationin our case); thus

$$g = C \circ \lambda,$$

with $C \in \mathrm{PSL}(2, \mathbb{C})$. It remains to evaluate C.

Evaluting the last equation at P_0, we see that C fixes ∞. Thus there are constants $\alpha \neq 0$ and β such that

$$g = \alpha\lambda + \beta.$$

It is easiest to proceed by translating this equation to one involving theta constants:

$$\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}(0,\cdot)=\alpha\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}(0,\cdot)+\beta\theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\cdot).$$

By equating leading terms of the Fourier series expansions of these functions we see that

$$\alpha=-1 \quad\text{and}\quad \beta=1.$$

Thus we conclude that

$$1=\lambda+g$$

and

$$\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}(0,\cdot)+\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}(0,\cdot)=\theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\cdot).$$

We have therefore proven one half of the following

THEOREM 6: *For all points τ in $\mathbb{H}^2$ the following two identities hold:*

$$\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}(0,\tau)+\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}(0,\tau)=\theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)$$

and

$$\exp(\pi\imath/3)\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}(0,\tau)+\exp(2\pi\imath/3)\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}(0,\tau)=\theta^3\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}(0,\tau).$$

Proof: The first identity has been derived above; the proof of the second one is similar. ∎

Remark 8: A similar argument establishes the well known classical identity among the three theta constants with even (integer) characteristic:

$$\theta^4\begin{bmatrix}0\\ 0\end{bmatrix}=\theta^4\begin{bmatrix}0\\ 1\end{bmatrix}+\theta^4\begin{bmatrix}1\\ 0\end{bmatrix}.$$

See the end of [1] for a more standard proof. ∎

Till now we have been concentrating on the four characteristic classes which are fixed pointwise by $\Gamma(3)$. Representatives for these can be taken to be

$$\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix},\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix},\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix},\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}.$$

As in the introduction we shall label these characteristics by the integers

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \sim 1, \quad \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \sim 2, \quad \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} \sim 3, \quad \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \sim 4.$$

We recall the generators B and A of the modular group. In the usual notation of permutations, $\eta(B)$, the permutation induced by B is $(1\ 3)(1\ 2)$ and $\eta(A) = (1\ 3)(2\ 4)$. These two even permutations generate the alternating group on four elements, $\mathcal{A}_4$. We have thus shown (this is well known) that $\Gamma/\Gamma(3)$ is isomorphic to $\mathcal{A}_4$. As a matter of fact,

$$\eta(B) = (1\ 3)(1\ 2),\ \eta(A) = (1\ 3)(2\ 4),\ \eta(B^2) = (1\ 2)(1\ 3),\ \eta(B \circ A) = (2\ 3)(2\ 4),$$

$$\eta(A \circ B) = (1\ 2)(1\ 4),\ \eta(B^2 \circ A) = (1\ 4)(1\ 2),\ \eta(A \circ B \circ A) = (1\ 4)(1\ 3),$$

$$\eta(B \circ A \circ B^2) = (1\ 2)(3\ 4),\ \eta(B^2 \circ A \circ B) = (1\ 4)(2\ 3),$$

$$\eta(A \circ B^2) = (2\ 4)(2\ 3),\ \text{and}\ \eta(A \circ B^2 \circ A) = (1\ 3)(1\ 4).$$

We have already seen that we have identities among the third powers of the four theta constants with characteristics in the above list. We now show how to use these identities to obtain $\mathcal{A}_4$ as a group of fractional linear transformations, and, more importantly, to obtain a modular invariant which generalizes the classical λ function. To simplify notation, we let

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = \theta_i,$$

provided we have identified the characteristic $\left[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix}\right]$ with the integer (index) i. Here i ranges over the integers 1, 2, 3, and 4. In this notation (after some trivial substitutions), the identities we have derived are:

$$\theta_3^3 = \theta_2^3 - \theta_1^3 \tag{17}$$

and

$$\theta_4^3 = \epsilon^2\theta_2^3 + \theta_1^3, \tag{18}$$

where $\epsilon = \exp \pi\imath/3$. Recall the definition

$$\lambda(\tau) = \frac{\theta_1^3}{\theta_2^3}, \quad \tau \in \mathbb{H}^2.$$

We have already seen that this function provides us with a conformal homeomorhism of $\mathbb{H}^2/\Gamma(3)$ on a four times punctured sphere. One easily computes the punctures to be 0, 1, ϵ^5, and ∞. We know that $\Gamma/\Gamma(3)$ is isomorphic to $\mathrm{Aut}(\mathbb{H}^2/\Gamma(3))$, the group of conformal automorphisms of the Riemann surface $\mathbb{H}^2/\Gamma(3)$. As a mater of fact we have a surjective homomorphism

$$\Pi\colon \Gamma \to \mathrm{Aut}(\mathbb{H}^2/\Gamma(3))$$

with kernel $\Gamma(3)$ defined by

$$\lambda \circ \gamma = \Pi(\gamma) \circ \lambda.$$

For $k = 2$, 3, and 5, $\mathbb{H}^2/\Gamma(k)$ is a punctured sphere and hence $\mathrm{Aut}(\mathbb{H}^2/\Gamma(k))$ is a group of Möbius transformations. We proceed to describe the homomorphism Π for the case $k = 3$.

As already remarked, the group Γ operates (on the right) on the function λ by substitution. Since the action of $\Gamma(3)$ is trivial, we actually have an action of $\Gamma/\Gamma(3)$ on λ. We choose (in the table below) a set of representatives for the nontrivial cosets of $\Gamma/\Gamma(3)$ and use (7) and (6) to obtain

$$\lambda \circ B = \pm\frac{\theta_3^3}{\theta_1^3}, \quad \lambda \circ A = \pm\epsilon^5\frac{\theta_3^3}{\theta_4^3}, \quad \lambda \circ (B^2) = \pm\frac{\theta_2^3}{\theta_3^3}, \quad \lambda \circ (B \circ A) = \pm\epsilon^5\frac{\theta_1^3}{\theta_3^3},$$

$$\lambda \circ (A \circ B) = \pm\epsilon^4\frac{\theta_2^3}{\theta_4^3}, \quad \lambda \circ (B^2 \circ A) = \pm\epsilon^5\frac{\theta_4^3}{\theta_1^3},$$

$$\lambda \circ (A \circ B \circ A) = \pm\epsilon^4\frac{\theta_4^3}{\theta_2^3}, \ \lambda \circ (B \circ A \circ B^2) = \pm\epsilon^5\frac{\theta_2^3}{\theta_1^3}, \ \lambda \circ (B^2 \circ A \circ B) = \pm\frac{\theta_4^3}{\theta_3^3},$$

$$\lambda \circ (A \circ B^2) = \pm\frac{\theta_1^3}{\theta_4^3}, \quad \text{and} \quad \lambda \circ (A \circ B^2 \circ A) = \pm\epsilon^2\frac{\theta_3^3}{\theta_2^3}.$$

We need to eliminate the ambiguous signs and to express the right hand sides of the above equations in terms of λ. We use the identities among the cubes of the theta constants (17) and (18), the fact that $\Pi(\gamma)$ has finite order for all $\gamma \in \Gamma$ (hence the trace of each of these fractional linear transformations lies in the open real interval $(-2, 2)$), and (when needed) the fact that the Möbius transformation $\Pi(\gamma)$ permutes the four points in $\{\infty,\ 0,\ 1,\ \epsilon^5\}$. We compute

$$\lambda \circ B = \frac{\lambda - 1}{\lambda}, \quad \lambda \circ A = \frac{\lambda - 1}{\epsilon\lambda - 1}, \quad \lambda \circ (B^2) = \frac{1}{1 - \lambda}, \quad \lambda \circ (B \circ A) = \frac{\epsilon^5\lambda}{\lambda - 1},$$

$$\lambda \circ (A \circ B) = \frac{\epsilon}{\lambda + \epsilon^2}, \quad \lambda \circ (B^2 \circ A) = \frac{\epsilon\lambda - 1}{\epsilon^2\lambda},$$

$$\lambda\circ(A\circ B\circ A)=1-\epsilon\lambda,\ \lambda\circ(B\circ A\circ B^2)=\frac{1}{\epsilon\lambda},\ \lambda\circ(B^2\circ A\circ B)=\frac{\lambda+\epsilon^2}{\lambda-1},$$

$$\lambda\circ(A\circ B^2)=\frac{\epsilon\lambda}{\epsilon\lambda-1},\quad \text{and}\quad \lambda\circ(A\circ B^2\circ A)=\epsilon^5(1-\lambda).$$

The above list produces 11 holomorphic functions on $\mathbb{H}^2$. We label them $\{f_1, \ldots, f_{11}\}$. We label the "identity function" $\tau\mapsto\lambda(\tau)$ by f_{12}. We form for $n=1, 2, 3, \ldots$

$$\varphi_n=\sum_{i=1}^{12} f_i^n.$$

Theorem 7: *The elementary symmetric functions (more generally any symmetric polynomial) of the 12 functions defined above are invariant under the modular group. In particular,*

$$\varphi_1=4(1-\epsilon^2),$$

$$\varphi_2=-4\epsilon^2,$$

and

$$\varphi_3=3\left(\frac{\lambda(\lambda-1)(\epsilon\lambda-1)}{\epsilon}-\frac{(\lambda-1)(\epsilon\lambda-1)}{\lambda^3}+\frac{\epsilon\lambda(\epsilon\lambda-1)}{(\lambda-1)^3}-\frac{\lambda(\lambda-1)}{(\epsilon\lambda-1)^3}\right)$$

is a nontrivial modular function with a simple pole at $\imath\infty$ (in fact φ_3 is a branched holomorphic universal cover of the orbifold $\mathbb{H}^2/\Gamma$ and defines a 12 sheeted cover of $\mathbb{H}^2/\Gamma$ by $\mathbb{H}^2/\Gamma(3)$).

Proof: Note that λ is invariant under $\Gamma(k)$ and

$$\varphi_n=\sum_{\Gamma(k)\backslash\Gamma}\lambda^n\circ\gamma$$

is invariant under Γ. Calculations yield the formulae for the sums of the first three powers of λ. The proof of the other assertions is routine. We are averaging a function of degree three on $\mathbb{H}^2/\Gamma(3)$ with respect to the group $\Gamma/\Gamma(3)$ of order 12. One expects a function of degree 36. Clearly, we have some cancelation. ∎

Remark 9: The function which sends $\tau\in\mathbb{H}^2$ to

$$\lambda+\lambda\circ B+\lambda\circ B^2=\frac{\lambda^3-3\lambda+1}{\lambda(\lambda-1)}$$

is automorphic for the group $G = \langle \Gamma(3),\ B \rangle$. This defines a three sheeted cover of $\mathbb{H}^2/G$ by $\mathbb{H}^2/\Gamma(3)$. This fact opens up a series problems that we will investigate in future papers. ∎

We finally turn our attention to the "complementary set" (consisting of classes represented by nonintegral characteristics of the form $\begin{bmatrix} m/3 \\ m'/3 \end{bmatrix}$ with not both m and m' odd) of twelve characteristics represented by

$$\begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} \sim 1, \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} \sim 2, \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \sim 3, \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \sim 4, \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \end{bmatrix} \sim 5, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \sim 6,$$

$$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \sim 7, \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} \sim 8, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \sim 9, \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix} \sim 10, \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} \sim 11, \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \sim 12.$$

The group $\Gamma(3)$ is generated by the motions

$$T_2\colon z \mapsto \frac{z}{3z+1}, \quad T_3\colon z \mapsto \frac{-2z-3}{3z+4}, \quad T_4\colon z \mapsto \frac{4z-3}{3z-2},$$

and $T_4 \circ T_2 \circ T_3 = T_1\colon z \mapsto z+3$. It acts on the above characteristic classes (by permutation). This defines a homomorphism from $\Gamma(3)$ to $\mathcal{S}_{12}$. It can be shown that $\Gamma(6)$ is the kernel of this homomorphism.

The permutation induced by T_2 is (2 12)(3 11)(4 10)(5 9) while the permutation induced by T_3 is (1 12)(3 7)(4 6)(5 8). It is not necessary to find the permutation induced by T_4 since $T_3 \circ T_4^{-1}\colon z \mapsto (13z-18)/(-18z+25) \in \Gamma(6)$, T_4 will induce the same permutation as T_3. The above permutations induced by T_2 and T_3 generate a subgroup G of order 6 of $\mathcal{S}_{12}$ (the image of $\Gamma(3)$ under the homomorphism).

If we denote by Y, the space of 12 characteristic classes in our list, then

$$Y/G = \{a = [1,1,12,12,2,2],\ b = [3,11,7,11,7,3],$$
$$c = [4,10,6,10,4,6],\ d = [5,9,8,9,8,5]\}$$

has four elements and Γ acts on Y/G. We check the action of Γ on these four elements. We easily find that B induces the permutation $(d\ c)(d\ b)$ and that and that A induces the permutation $(a\ b)(c\ d)$. If we now compare this with the action of Γ on the four (fractional) characteristic classes fixed pointwise by $\Gamma(3)$, we find that we should associate

the characteristic class of $\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$ with the orbit $[1,1,2,2,12,12]$, the characteristic class of $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ with the orbit [5,5,8,8,9,9],

the characteristic class of $\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}$ with the orbit [4,10,6,10,4,6],

and

the characteristic class of $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ with the orbit [3,3,7,7,11,11].

The above arguments suggest the following theorem which gives a set of proportionalities with a different flavor than the ones obtained for $k = 2$.

THEOREM 8: *The quotient of any two of the following products is a constant on the upper half plane:*

$$\theta\begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}\theta\begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}, \quad \theta\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\theta\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\theta\begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix},$$

$$\theta\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}\theta\begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}\theta\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \theta\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \end{bmatrix}\theta\begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}\theta\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Proof: We consider the 24-th powers of the above products. Each is a multiplicative (associated to some character (homomorphism of $\Gamma(3)$ into the unit circle)) automorphic 18-form, say φ. The ratio of two of these is hence a multiplicative automorphic function f for the group $\Gamma(3)$. The singularities and zeros of the projection F of f to $\mathbb{H}^2/\Gamma(3)$ are at the punctures. The form φ has reduced order -9 at each of the punctures. The (multivalued) multiplicative function F is hence regular at the punctures (thus the character associated with F must be trivial). We conclude that F is a (single valued) holomorphic function on the compactification of $\mathbb{H}^2/\Gamma(3)$ and hence constant. ∎

11. $k = 5$

We would now like to repeat as much of the above discussion for the group $\Gamma(5)$. As a maximal set of inequivalent parabolic fixed points for this group, we may take the following 12 points:

$$\{-2, -3/2, -1, -1/2, \infty, 0, 2/5, 1/2, 1, 3/2, 2, 5/2\}.$$

Calculations (using Mathematica) show, for example, that

$$\left(\Phi\begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}\right) = \frac{P_{-1}^{15} P_{3/2}^{15}}{P_\infty^9 P_0^9 P_1 P_2 P_{-2}^9 P_{1/2} P_{5/2} P_{-1/2}^9 P_{-3/2}^9 P_{2/5}}.$$

Similar formulae hold for the other 11 characteristics.

Fifth powers of ratios of theta constants give us meromorphic functions of either degree 7 or degree 5 on the 12-punctured sphere $\mathbb{H}^2/\Gamma(5)$. The divisors of these degree 5 functions are supported at the precisely 10 of the 12 punctures. Each of these has 5 simple zeros and 5 simple poles. We illustrate with

$$\left(\frac{\theta^5\begin{bmatrix}\frac{1}{5}\\\frac{1}{5}\end{bmatrix}}{\theta^5\begin{bmatrix}\frac{3}{5}\\\frac{3}{5}\end{bmatrix}}\right) = \frac{P_1P_2P_{1/2}P_{5/2}P_{2/5}}{P_\infty P_0 P_{-2}P_{-1/2}P_{-3/2}}.$$

Let us call the function produced above f. Observe that f is regular and nonzero at the two punctures P_{-1} and $P_{3/2}$. Riemann-Hurwitz tells us that f has total branch number 8. Let us denote $f(P_{-1})$ by ω_1 and $f(P_{3/2})$ by ω_2. Simple calculations show that

$$\omega_1 = \frac{\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}}{\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}}(0, i\infty) = \frac{\cos^5\frac{\pi}{10}}{\cos^5\frac{3\pi}{10}} \quad \text{and} \quad \omega_2 = \frac{\theta^5\begin{bmatrix}1\\\frac{7}{5}\end{bmatrix}}{\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}}(0, i\infty) = \frac{\cos^5\frac{7\pi}{10}}{\cos^5\frac{\pi}{10}}$$

and that the branch numbers of f at P_{-1} and $P_{3/2}$ must be ≥ 4. It follows that the branch number of f at each of these punctures is precisely four and that f is unbranched elsewhere. Hence a fifth root g of

$$\frac{f-\omega_1}{f-\omega_2}$$

is a holomorphic universal covering map of the 12-punctured sphere $\mathbb{H}^2/\Gamma(5)$ with a simple zero at -1 and a simple pole $3/2$. By considering several degree 5 maps, one can once again produce relations among theta constants.

We have therefore proven most of

THEOREM 9: *The 5 single valued functions $((f-\omega_1)/(f-\omega_2))^{1/5}$ are holomorphic universal coverings of the Riemann surface $S = \mathbb{H}^2/\Gamma(5)$. Moreover, there exist complex numbers $\omega_3 \neq \omega_4$ such that the functions*

$$\tau \mapsto \left(\frac{\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}(0,\tau) - \omega_3}{\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}(0,\tau) - \omega_4}\right)^{\frac{1}{5}}$$

are also holomorphic universal coverings of S by $\mathbb{H}^2$ and their fifth powers defines a degree 5 map of S onto the 4-punctured sphere given as the quotient of S by the automorphism of this surface induced by B.

Proof: The first statement has been proved before the statement of the theorem. The proof of the second statement follows as does the first after one checks the respective divisors of the functions. The last statement is also an immediate consequence of the structure of the divisors. ∎

Remark 10: Instead of the fifth powers of the last function, we could use $\theta^5\begin{bmatrix}1\\1/5\end{bmatrix}(0,\tau)/\theta^5\begin{bmatrix}1\\3/5\end{bmatrix}(0,\tau)$. ∎

12. Concluding remarks

In this paper we have attempted to generalize what we consider as the classical theory (the $k=2$ case) to the principal congruence subgroups of the modular group of odd prime level. We have concentrated on explicitly writing formulae for covering maps which generalize the classical λ function of the $k=2$ theory. We have also attempted to lay the foundation for further work we intend to do which will generalize the section on $k=2$ of this paper and the result we obtained in the section on $k=3$. We have omitted some very interesting connections of this theory to the theory of partitions and Ramanujan congruences. We hope in a subsequent publication to discuss many of the things that time did not permit us to do here.

References

[1] H. M. Farkas and I. Kra, *Riemann Surfaces* (second edition), Graduate Texts in Mathematics, Vol. 71, Springer-Verlag, Berlin, 1992.

[2] R. C. Gunning, *Lectures on Modular Forms,* Annals of Mathematics Studies, Vol. 48, Princeton University Press, 1962.

[3] I. Kra, *Automorphic Forms and Kleinian Groups,* Benjamin, New York, 1972.

[4] H. E. Rauch and H. M. Farkas, *Theta Functions with Applications to Riemann Surfaces,* Williams and Wilkins, Baltimore, 1974.

[5] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions,* Kano memorial lectures, Vol. 1, Iwannami Shoten, Publisher and Princeton University Press, 1971.